Directions: Solve the following problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. Dirichlet product and Mobius pairs. Let $\mathbb{O}\colon \mathbb{Z}^+ \to \mathbb{R}$ be the identically zero function.
   (a) Prove that the Dirichlet product is bilinear. That is, for all functions $f, g, h\colon \mathbb{Z}^+ \to \mathbb{R}$ and all constants $\alpha, \beta \in \mathbb{R}$, we have $f * (\alpha g + \beta h) = \alpha(f * g) + \beta(f * h)$.
   (b) Prove that if $f * g = \mathbb{O}$, then $f = \mathbb{O}$ or $g = \mathbb{O}$.
   (c) Show that if both (h_1, h_2) and (h_2, h_1) are Mobius pairs, then $h_1 = h_2 = \mathbb{O}$.
2. *Primitive Roots I.*
   (a) Find all primitive roots modulo 5, modulo 9, modulo 11, modulo 13, and modulo 15.
   (b) Let a and m be positive, relatively prime integers. Let S be the set of primes dividing $\phi(m)$. Prove that if $a^{\phi(m)/p} \not\equiv 1 \pmod{m}$ for each $p \in S$, then a is a primitive root of m.
3. *Primitive Roots II.*
   (a) Let m_1 and m_2 be relatively prime integers, and suppose that p and q are odd primes such that $p \mid m_1$ and $q \mid m_2$. Let $m = m_1 m_2$. Prove that if a and m are relatively prime, then $a^{\phi(m)/2} \equiv 1 \pmod{m}$.
   (b) Show that if m has two distinct odd prime divisors, then m has no primitive roots.
4. [NT 8-1.4] Modify the proof of Theorem 8–1 to prove that there exist infinitely many primes congruent to $5 \pmod{6}$.
5. [NT 8-1.16] Let $n = 132!$. How many zeros are at the end of the base 2 representation of n? How many zeros are at the end of the base 10 representation of n?
6. *Upper bound on* $\sum_{p \le n} \frac{1}{p}$. Let $P[a, b)$ be the set of all primes p such that $a \le p < b$. Let $H_n = \sum_{k=1}^{n} \frac{1}{k}$ and recall $\ln n \le H_n \le 1 + \ln n$.
   (a) Show that $\sum_{p \in P[1,2^t)} \frac{1}{p} \le 16 H_t$. (Hint: use Chebychev's theorem to bound $\sum_{p \in P[2^{k-1},2^k)} \frac{1}{p}$.)
   (b) Prove that $\sum_{p \le n} \frac{1}{p} \le C \ln\ln n$ for some constant C.
7. *Lower bound on* $\sum_{p \le n} \frac{1}{p}$. Let $z_n = \sum_{p \le n} \frac{1}{p}$ and let $H_n = \sum_{k=1}^{n} \frac{1}{k}$.
   (a) Use an integral comparison to show that $\sum_{k \ge t} \frac{1}{k^2} \le \frac{1}{t-1}$. Conclude that $\sum_p \frac{1}{p^2} \le \frac{3}{4}$.
   (b) Prove that $e^{z_n} \ge \prod_{p \le n}(1 + \frac{1}{p})$.
   (c) Prove that $\prod_{p \le n}(1 + \frac{1}{p}) \ge (1 - \sum_{p \le n} \frac{1}{p^2}) H_n$. Conclude that $z_n \ge (1 - o(1)) \ln\ln n$.
8. **[Challenge]** Let $f(x)$ be a polynomial with integer coefficients. Prove that for infinitely many primes p, there exists an integer n such that $p \mid f(n)$.