Directions: Solve the following problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. Let M be the set of all positive integers m such that $a^{\phi(m)+1} \equiv a \pmod{m}$ for each integer a. Give a simple characterization of M and prove that your characterization is correct.

2. [NT 5-2.{9,10}]

   (a) Prove that if p is a prime and $p \equiv 1 \pmod 4$, then $\left[\left(\frac{p-1}{2}\right)!\right]^2 \equiv -1 \pmod p$.

   (b) Use the above to find a solution for each of the following.

      i. $x^2 \equiv -1 \pmod{13}$

      ii. $x^2 \equiv -1 \pmod{17}$

3. Prove that if m is not a prime and $\phi(m) \mid m - 1$, then m has at least three distinct prime factors.

4. Prove that the number of ways of writing n as a sum of consecutive positive integers equals $d(m)$, where m is the largest odd divisor of n. For example, for $n = 42$ we have $m = 21$ and $d(m) = d(21) = 4$. As expected, there are 4 ways to express 42 as the sum of consecutive positive integers: 42, $13 + 14 + 15$, $9 + 10 + 11 + 12$, and $3 + 4 + \cdots + 9$.

5. [NT 6-4.2] Prove that if $f(n)$ is multiplicative and not identically zero, then $\sum_{d|n} \mu(d) f(d) = \prod_{p|n} (1 - f(p))$.

6. **[Challenge]** Prove that if n divides $3^n - 1$, then $n = 1$ or n is even.