Directions: Solve the following problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. Prove that p is the smallest prime that divides $(p-1)!+1$.

2. [NT 5-2.4] Let $k = \phi(m)$. Prove that if $r_1, \ldots, r_k$ is a reduced residue system modulo m and m is odd, then $r_1 + r_2 + \cdots + r_k \equiv 0 \pmod{m}$.

3. [NT 5-3.4]

   (a) Prove that for each n, there are n consecutive integers, each of which is divisible by a perfect square larger than 1.

   (b) Using your proof above, explicitly find 3 consecutive integers, each of which is divisible by a perfect square larger than 1. In your answer, give the integers as well as the corresponding perfect squares.

4. [NT 5-4.1] Find the set of solutions to the following system of congruences:

$$\begin{aligned} 2x &\equiv 1 \pmod{5} \\ 3x &\equiv 9 \pmod{6} \\ 4x &\equiv 1 \pmod{7} \\ 5x &\equiv 9 \pmod{11} \end{aligned}$$

5. Let $A = \{a^2 - b^2 \colon\ a, b \in \mathbb{Z}\}$. Give a simple characterization of A (with proof of correctness).

6. Prove that there are infinitely many integers n such that $n \mid 2^n + 1$. Hint: first, find the three smallest such integers.

7. [**Challenge**] Prove that if $n \geq 2$, then the sum $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is not an integer.