Directions: Solve 3 of the following 4 problems. All written work must be your own, using only permitted sources. See the "General Guidelines and Advice" on the homework page for more details.

1. Prove that if G has no induced $2K_2$, then $\chi(G) \le \binom{\omega(G)+1}{2}$. Here, $2K_2$ denotes the disjoint union of two copies of K_2; that is, $2K_2$ is the 1-regular 4-vertex graph. (Hint: use a maximum clique to define a collection of $\binom{\omega(G)}{2} + \omega(G)$ independent sets that cover the vertices.)

2. Let G be a k-chromatic graph with girth 6 and order n. Construct G' as follows. Let T be an independent set of kn vertices. Take $\binom{kn}{n}$ pairwise disjoint copies of G, one for each way to choose an n-set $S \subset T$. Add a matching between each copy of G and its corresponding n-set S. Prove that the resulting graph has chromatic number $k+1$ and girth 6. (Comment: Since C_6 has chromatic number 2 and girth 6, the process can start and these graphs exist.)

3. Let G be a graph with no induced copy of the claw ($K_{1,3}$).

   (a) Show that in a proper coloring, each subgraph of G induced by the union of two color classes consists of paths and even cycles.

   (b) An *equitable* coloring of a graph is a proper vertex-coloring in which every pair of color classes differs in size by at most 1. Prove that G has an equitable coloring that is optimal (i.e. uses just $\chi(G)$ colors).

4. Let G be a connected n-vertex planar graph with m edges and girth g. Prove that $m \le (n-2)g/(g-2)$.

5. Prove that every n-vertex plane graph decomposes into at most $2n-4$ edges and facial triangles. Hint: induction on the number of facial triangles.

6. *Triangles and distance in planar graphs.*

   (a) Prove that if G is an n-vertex m-edge plane graph, then G has at least $8+2m-4n$ facial triangles.

   (b) Prove that if G is an n-vertex planar graph and $\delta(G) \ge 5$, then G contains K_4^- as a subgraph. (Here, K_4^- is the graph obtained from K_4 by deleting an edge; it is sometimes called the *kite*.)

   (c) Construct a sequence of 4-regular planar graphs such that in the kth graph, the distance between any two triangles is at least k.

   Remark: From part (b), a planar graph with minimum degree at least 5 contains a pair of triangles that share an edge and so are at distance zero. Therefore there is no analogous construction of 5-regular planar graphs with well-separated triangles.