Directions: Solve 5 of the following 6 problems. All written work must be your own, using only permitted sources. See the "General Guidelines and Advice" on the homework page for more details.

1. Let $d_1, \ldots, d_n$ be positive integers with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, \ldots, d_n$ if and only if $\sum_{i=1}^{n} d_i = 2n - 2$.

2. For $n \geq 4$, let G be an n-vertex graph with at least $2n-3$ edges. Prove that G has two cycles of equal length.

3. Determine with proof $\mathrm{ex}(n, P_n)$, the maximum number of edges in an n-vertex graph that does not contain a spanning path.

4. (a) Prove that every connected graph has an orientation in which the number of vertices with odd outdegree is at most 1. (Hint: consider an orientation with the fewest number of vertices with odd outdegree.)

   (b) Use part (a) to conclude that every connected graph with an even number of edges has a P_3-decomposition.

5. Let G be a directed graph without loops. Prove that G has an independent set S such that every vertex in G is reachable from a vertex in S by a directed path of length at most 2. Hint: use induction on $|V(G)|$ and recall that the induction hypothesis applies to all graphs with fewer vertices, not just graphs with $|V(G)| - 1$ vertices.

6. *Counting in tournaments.* Let T be an n-vertex tournament.

   (a) Prove that T has $\binom{n}{3} - \sum_{v \in V(T)} \binom{d^+(v)}{2}$ (directed) 3-cycles.

   (b) For odd n, prove that there is an n-vertex Eulerian tournament.

   (c) For odd n, determine the maximum possible number of 3-cycles in an n-vertex tournament.