Directions: Solve 5 of the following 6 problems. All written work must be your own, using only permitted sources. See the "General Guidelines and Advice" on the homework page for more details.

1. Graphs defined algebraically.
   (a) Let G_n be the graph whose vertices are the permutations of $\{1, \ldots, n\}$ with two permutations adjacent if they differ by interchanging a pair of adjacent entries. Note that $G_3 = C_6$. Prove that G_n is connected.
   (b) Let H_n be the graph whose vertices are the permutations of $\{1, \ldots, n\}$ with two permutations adjacent if they differ by interchanging a pair of entries (which may or may not be adjacent). Note that $H_3 = K_{3,3}$ and G_n is a subgraph of H_n. Prove that H_n is bipartite. Hint: for each permutation $a_1 \cdots a_n$, count the pairs (i, j) with $i < j$ and $a_i > a_j$; these are called *inversions*.

2. Forbidden induced subgraphs.
   (a) Let G be a connected graph not having P_4 or C_3 as an induced subgraph. Prove that G is a biclique.
   (b) Let G be a connected graph not having P_4 or C_4 as an induced subgraph. Prove that G has a vertex adjacent to all other vertices. (Hint: consider a vertex of maximum degree.)

3. Let P and Q be paths of maximum length in a connected graph G. Prove that P and Q have a common vertex.

4. An xy-trail W is *greedy* if every edge incident to y is contained in W. Let G be an Eulerian graph, and let x be a vertex in G. Prove that every greedy trail starting from x is an Eulerian circuit if and only if every cycle in G contains x.

5. Cut-edges and degrees.
   (a) Prove that an even graph has no cut-edge. For each $k \geq 1$, construct a $(2k+1)$-regular graph having a cut-edge.
   (b) For $k \geq 2$, prove that a k-regular bipartite graph has no cut-edge.

6. Counting cycles.
   (a) Count the cycles of length n in K_n.
   (b) Count the cycles of length $2n$ in $K_{n,n}$.
   (c) Count the cycles of length 5 in the Petersen graph.