Solutions

Math261 **Worksheet 9** **October 28, 2014**

1. Write the following functions as a product of two trigonometric functions.

   (a) $\cos(2t) + \cos(3t)$

①
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin(\alpha)\sin(\beta)$$
$$\cos(\alpha-\beta) = \cos(\alpha)\cos(-\beta) - \sin(\alpha)\sin(-\beta)$$
$$= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta).$$

$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos(\alpha)\cos(\beta).$$

②
$$\alpha+\beta = 2t$$
$$\alpha-\beta = 3t$$
$$2\alpha = 5t$$
$$\alpha = \frac{5}{2}t$$

③
$$\beta = 2t - \alpha$$
$$= \frac{4}{2}t - \frac{5}{2}t$$
$$= -\frac{1}{2}t$$

④ So:

$$\cos(2t) + \cos(3t)$$
$$= 2\cos\left(\frac{5}{2}t\right)\cos\left(-\frac{1}{2}t\right)$$
$$= \boxed{2\cos\left(\frac{5t}{2}\right)\cos\left(\frac{t}{2}\right)}$$

   (b) $\cos(2t) + \sin(3t)$. Hint: $\cos(z) = \sin(z + \pi/2)$ and $\sin(z) = \cos(z - \pi/2)$.

①
$$= \cos(2t) + \cos\left(3t - \frac{\pi}{2}\right)$$
$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos(\alpha)\cos(\beta)$$

$$\alpha+\beta = 2t$$
$$\alpha-\beta = 3t - \frac{\pi}{2}$$
$$2\alpha = 5t - \frac{\pi}{2}$$
$$\alpha = \frac{5}{2}t - \frac{\pi}{4}$$
$$\beta = 2t - \alpha = 2t - \left(\frac{5}{2}t - \frac{\pi}{4}\right)$$
$$= -\frac{1}{2}t + \frac{\pi}{4}$$

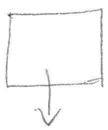
② So $\cos(2t) + \sin(3t)$

$$= 2\cos\left(\frac{5}{2}t - \frac{\pi}{4}\right)\cos\left(-\frac{1}{2}t + \frac{\pi}{4}\right)$$
$$= 2\cos\left(\frac{5}{2}t - \frac{\pi}{4}\right)\cos\left(\frac{t}{2} - \frac{\pi}{4}\right)$$

2. [3.8.9] An undamped spring-mass system with a mass that weighs 6 lb and a spring constant of 1 lb/in is suddenly set in motion at $t = 0$ by an external force of $4\cos 7t$ lb. Determine the position of the mass $u(t)$ (in) at time t (s), expressing $u(t)$ ~~$= u_c(t) + U(t)$, where $u_c(t)$ is the transient solution and the steady state solution $U(t)$ is expressed~~ as a product of two trigonometric functions.

Units: in, s; force: lb.

① $mu'' + \gamma u' + ku = 4\cos 7t$



w=mg

$6\,\text{lb} = m\, 32\frac{\text{ft}}{\text{s}^2}$

$6\,\text{lb} = m \cdot 32\frac{\text{ft}}{\text{s}^2} \cdot \frac{12\,\text{in}}{\text{ft}}$

$m = \frac{6}{32\cdot 12}\ \frac{\text{lb s}^2}{\text{in}}$

$= \frac{1}{64}\ \frac{\text{lb s}^2}{\text{in}}$

$\gamma = 0$ (undamped)

$k = 1\ \frac{\text{lb}}{\text{in}}$

$\frac{1}{64}u'' + u = 4\cos 7t, \quad u(0)=0, \ u'(0)=0$

② Transient Soln:

$\frac{1}{64}r^2 + 1 = 0$

$r^2 = -64$

$r = \pm 8i$

$u_c(t) = c_1\cos(8t) + c_2\sin(8t)$

③ Steady state:

$\left[U(t) = A\cos(7t) + B\sin(7t)\right]\cdot 1$

$\left[U'(t) = 7B\cos(7t) - 7A\sin(7t)\right]\cdot 0$

$\left[U''(t) = -49A\cos(7t) - 49B\sin(7t)\right]\cdot \frac{1}{64}$

$\left(A - \frac{49}{64}A\right)\cos(7t) + \left(B - \frac{49}{64}B\right)\sin(7t) = 4\cos(7t)$

$A - \frac{49}{64}A = 4, \qquad B - \frac{49}{64}B = 0$

$\qquad\qquad\qquad\qquad B = 0$

$\frac{15}{64}A = 4$

$A = \frac{256}{15}$

④ $u(t) = c_1\cos(8t) + c_2\sin(8t) + \frac{256}{15}\cos(7t)$

$u(0)=0$: $\quad 0 = c_1 + 0 + \frac{256}{15}, \quad c_1 = \frac{-256}{15}$

$u'(t) = -8c_1\sin(8t) + 8c_2\cos(8t) - \frac{256\cdot 7}{15}\sin(7t)$

$u'(0)=0$: $\quad 0 = 0 + 8c_2 + 0, \quad c_2 = 0$

⑤ $u(t) = \frac{256}{15}\left(\cos(7t) - \cos(8t)\right)$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$-\ \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin\alpha\sin\beta$

$\cos(\alpha-\beta) - \cos(\alpha+\beta) = 2\sin\alpha\sin\beta$

$\alpha - \beta = 7t \qquad \beta = 8t - \alpha$

$\alpha + \beta = 8t \qquad\ = \frac{16}{2}t - \frac{15}{2}t$

$2\alpha = 15t \qquad\ = \frac{1}{2}t$

$\alpha = \frac{15}{2}t$

$u(t) = \frac{256}{15}\left[2\sin\left(\frac{15}{2}t\right)\sin\left(\frac{1}{2}t\right)\right]$

$\boxed{u(t) = \frac{512}{15}\sin\left(\frac{15}{2}t\right)\sin\left(\frac{1}{2}t\right)}$

3. Now, modify the system in #2 so that the damping constant is $\gamma = 1\text{lb}\cdot\text{s/in}$. Find the forced response $U(t)$, expressed in the form $U(t) = R\cos(7t)$. How does this compare with the static response? Is this system ~~demonstrating~~ experiencing resonance?

① $\frac{1}{64}u'' + u' + u = 4\cos(7t)$

② $\frac{1}{64}r^2 + r + 1 = 0$

$$r = \frac{-1 \pm \sqrt{1 - 4\cdot\frac{1}{64}\cdot 1}}{\frac{1}{32}}$$

$$= -32 \pm 32\sqrt{1 - \frac{1}{16}}$$

$$= -32 \pm 8\sqrt{15}, \quad r \approx -62.98,\ -1.02$$

Notice: no i here!

$u_c(t) = c_1 e^{-62.98t} + c_2 e^{-1.02t}$ over damped system.

③ $[U(t) = A\cos(7t) + B\sin(7t)]\cdot 1$

$[U'(t) = 7B\cos(7t) - 7A\sin(7t)]\cdot 1$

$[U''(t) = -49A\cos(7t) - 49B\sin(7t)]\ \frac{1}{64}$

$$\left[\left(1 - \frac{49}{64}\right)A + 7B\right]\cos(7t) + \left[\left(1 - \frac{49}{64}\right)B - 7A\right]\sin(7t) = 4\cos(7t)$$

$\frac{15}{64}A + 7B = 4$

$-7A + \frac{15}{64}B = 0$

$7A + \frac{7^2\cdot 64}{15}B = \frac{4\cdot 7\cdot 64}{15}$

$B \approx 0.5708$

$A \approx 0.0191$

④ $U(t) \approx 0.0191\cos(7t) + 0.5708\sin(7t)$

$= R\cos(7t - \delta)$

$R = \sqrt{A^2 + B^2} \approx 0.5711$

$\delta = \tan^{-1}\left(\frac{B}{A}\right) \approx 1.5373$

$$\boxed{U(t) = 0.5711\cos(7t - 1.5373)}$$

The forced response has amplitude 0.5711 in.

The static response is

$$\frac{F}{k} = \frac{4\text{ lb}}{1\text{ lb/in}} = 4\text{ in.}$$

So, the forced response is much smaller than the static response.

The system is not ~~done~~ experiencing resonance

Alternatively:

$$\Gamma = \frac{\gamma^2}{mk} = \frac{1^2}{\frac{1}{64}\cdot 1} = 64$$

Since $\Gamma \geq 2$, the system is too damped to experience resonance.