Solutions

1. Rewrite $-2\cos(3t) - 3\sin(3t)$ in the form $R\cos(\omega_0 t - \delta)$.

$R = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13}$

$R\cos(\delta) = -2$

$R\sin(\delta) = -3$

$\left(\frac{-2}{R}, \frac{-3}{R}\right)$

$\tan\delta = \frac{\sin\delta}{\cos\delta} = 3/2$

~~arctan~~

$\arctan\left(\frac{3}{2}\right) = 0.983$

Correct phase:

$\delta = \arctan\left(\frac{3}{2}\right) + \pi$

≈ 4.124

$\boxed{\sqrt{13}\cos(3t - 4.124)}$

2. [3.7.13] A certain vibrating system satisfies the equation $u'' + \gamma u' + u = 0$. Find the value of the damping coefficient γ for which the quasi period of the damped motion is 50% greater than the period of the corresponding undamped motion.

Damped

$r^2 + \gamma r + 1 = 0$

$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4}}{2}$

Assume $\gamma^2 - 4 < 0$, or else system does not oscillate.

$r = \frac{-\gamma}{2} \pm \frac{(\sqrt{4-\gamma^2})\,i}{2}$

$u = e^{-\gamma/2}\left(c_1 \cos\left(\frac{\sqrt{4-\gamma^2}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{4-\gamma^2}}{2}t\right)\right)$

Quasi-frequency: $\frac{\sqrt{4-\gamma^2}}{2}$

Quasi-period: $\frac{2\pi}{\frac{\sqrt{4-\gamma^2}}{2}} = \frac{4\pi}{\sqrt{4-\gamma^2}}$

Undamped:

$r^2 + 1 = 0$

$r^2 = -1$

$r = \pm i$

$u = e^{ti} = \cos(t) + i\sin(t)$

$u = c_1\cos(t) + c_2\sin(t)$

Frequency: 1

Period: $\frac{2\pi}{1} = 2\pi$

Find γ such that $\frac{4\pi}{\sqrt{4-\gamma^2}} = (1.5)\cdot 2\pi$

$\frac{4}{3} = \sqrt{4-\gamma^2}$

$\frac{16}{9} = 4 - \gamma^2$

$\gamma^2 = 4\left(1 - \frac{4}{9}\right)$

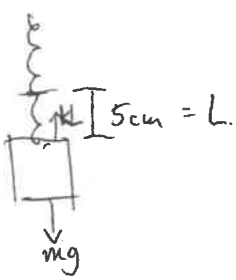
$\gamma = 2\sqrt{\frac{5}{9}}$

$= \boxed{\frac{2\sqrt{5}}{3}}$

3. [3.7.9] A mass of 20 g stretches a spring 5 cm. Suppose that the mass is also attached to a viscus damper with a damping constant of 400 dyn·s/cm. (Note: 1dyn = 1g cm/s^2). If the mass is pulled down an additional 2 cm and then released, (a) find its position u as a function of time t. (b) Determine the quasi-frequency and quasi-period. (c) Determine the ratio of the quasi-period to the period of corresponding undamped motion.

- $mu'' + \gamma u' + ku = 0$
- $m = 20\ g$
- $\gamma = 400\ \frac{dyn \cdot s}{cm} = 400 \cdot \frac{1g\ \cancel{cm}}{s^{\cancel{2}}} \cdot \frac{\cancel{s}}{\cancel{cm}}$

$= 400\ \frac{g}{s}$

- 

$mg = kL$

$(20g)\ 9.8\ \frac{m}{s^2} = k \cdot 5cm$

$(20g)\ 9.8 \cdot \frac{100\ \cancel{cm}}{s^2} = k \cdot 5\ \cancel{cm}$

$k = (4g) \cdot 9.8 \cdot 100 \cdot \frac{1}{s^2}$

$= 3920\ g/s^2$

- $20u'' + 400u' + 3920u = 0$

$20r^2 + 400r + 3920 = 0$

$r^2 + 20r + 196 = 0$

$r = \frac{-20 \pm \sqrt{400 - 4(196)}}{2}$

$= -10 \pm \sqrt{100 - 196}$

$= -10 \pm \sqrt{96}\ i$

$= -10 \pm 4\sqrt{6}\ i$

(a) $u = e^{-10t}\left(c_1 \cos(4\sqrt{6}\,t) + c_2 \sin(4\sqrt{6}\,t)\right)$

Impose $u(0) = 2$:

$2 = 1(c_1 \cdot 1 + c_2 \cdot 0)$; $c_1 = 2$

$u' = -10e^{-10t}\left(c_1 \cos(4\sqrt{6}\,t) + c_2 \sin(4\sqrt{6}\,t)\right)$
$+ e^{-10t}\left(-c_1 \sin(4\sqrt{6}\,t) \cdot 4\sqrt{6} + 4\sqrt{6}\,c_2 \cos(4\sqrt{6}\,t)\right)$

Impose $u'(0) = 0$:

$0 = -10 \cdot 1(c_1 + c_2 \cdot 0) + 1(0 + 4\sqrt{6}\,c_2 \cdot 1)$

$0 = -10c_1 + 4\sqrt{6}\,c_2$

$4\sqrt{6}\,c_2 = 10c_1 = 20$

$c_2 = \frac{5}{\sqrt{6}} = \frac{5\sqrt{6}}{6}$.

$u = \boxed{e^{-10t}\left(2\cos(4\sqrt{6}\,t) + \frac{5\sqrt{6}}{6}\sin(4\sqrt{6}\,t)\right)}$

(b) Quasi frequency: $4\sqrt{6}$ rad/sec

Quasi Period: $\frac{2\pi}{4\sqrt{6}} = \frac{\pi}{2} \cdot \frac{\sqrt{6}}{6} = \boxed{\frac{\sqrt{6}}{12}\pi}\ s$

(c) Undamped: $20u'' + 3920u = 0$

$20r^2 + 3920 = 0$

$r^2 = -196$, $r = \pm 14i$

$u = c_1 \cos(14t) + c_2 \sin(14t)$

Freq: 14 rad/sec Period: $\frac{2\pi}{14}$ sec

Ratio: $\frac{\frac{\sqrt{6}}{12}\cancel{\pi}}{\frac{2\cancel{\pi}}{14}} = \frac{7\sqrt{6}}{12} \approx \boxed{1.429}$