1. Convert the following system into a system of first order differential equations. Do not attempt to solve.

$$3x_1'' - 2x_1' + 5x_1 + 2x_2'' + x_2 = 0$$
$$x_2^{(3)} + x_1 + x_2' = 0$$

2. Find the general solution to the following.

   (a) $\mathbf{x}' = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} x$

   (b) $\mathbf{x}' = \begin{bmatrix} -4 & -9 & 3 \\ 0 & -1 & 0 \\ -6 & -18 & 5 \end{bmatrix} \mathbf{x}$

3. Find the Fourier series for $f(x) = x^2/2$ for $-2 \le x \le 2$ and $f(x+4) = f(x)$.

1. $y_1 = x_1$, $\underbrace{y_2 = y_1' = x_1'}_{\text{Eq 1}}$

$y_3 = x_2$, $\underbrace{y_4 = y_3' = x_2'}_{\text{Eq 2}}$, $\underbrace{y_5 = y_4' = x_2''}_{\text{Eq 3}}$

$3x_1'' - 2x_1' + 5x_1 + 2x_2'' + x_2 = 0$

$3y_2' - 2y_2 + 5y_1 + 2y_5 + y_3 = 0$ Eq 4

$x_2^{(3)} + x_1 + x_2' = 0$

$y_5' + y_1 + y_4 = 0$ Eq 5

Eq 1: $y_1' = y_2$

Eq 4: $y_2' = -\frac{5}{3}y_1 + \frac{2}{3}y_2 - \frac{1}{3}y_3 - \frac{2}{3}y_5$

Eq 2: $y_3' = y_4$

Eq 3: $y_4' = y_5$

Eq 5: $y_5' = -y_1 - y_4$

$$\boxed{y' = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{5}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & -\frac{2}{3} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 & 0 \end{bmatrix} y}$$

Note: other solns possible

2 (a) ① Eigenvals:

$$0 = \left|\begin{matrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{matrix}\right| = (3-\lambda)(1-\lambda) - (-1)(1)$$

$$= 3 - 4\lambda + \lambda^2 + 1$$

$$= \lambda^2 - 4\lambda + 4$$

$$= (\lambda - 2)^2$$

$\lambda = 2$, mult. 2.

② Eigenvectors

$$\begin{bmatrix} 3-2 & -1 \\ 1 & 1-2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\xi_1 - \xi_2 = 0 \qquad \xi = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}.$$

⇒ Need a second soln from generalized Eigenvector,

$$(A - \lambda I)\eta = \xi$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \eta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \left[\begin{matrix} 1 & -1 \\ 1 & -1 \end{matrix}\middle|\begin{matrix} 1 \\ 1 \end{matrix}\right] \rightsquigarrow \left[\begin{matrix} 1 & -1 \\ 0 & 0 \end{matrix}\middle|\begin{matrix} 1 \\ 0 \end{matrix}\right]$$

$$\eta_1 - \eta_2 = 1, \quad \eta_2 = c, \ \eta_1 = 1 + c$$

$$\eta = \begin{bmatrix} 1+c \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad \text{Choose } c = 0; \ \eta = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

So $x = (\xi t + \eta)e^{2t} = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) e^{2t}$ is our second soln

3 Gen Sol'n:

$$\boxed{x = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{2t}}$$

2b:

$$x' = \begin{bmatrix} -4 & -9 & 3 \\ 0 & -1 & 0 \\ -6 & -18 & 5 \end{bmatrix} x$$

1 Eigenvalues:

$$0 = \begin{vmatrix} -4-\lambda & -9 & 3 \\ 0 & -1-\lambda & 0 \\ -6 & -18 & 5-\lambda \end{vmatrix} = \left((-4-\lambda)(-1-\lambda)(5-\lambda) + 0 + 0 \right) - \left(3(-6)(-1-\lambda) + 0 + 0 \right)$$

$$= (-1-\lambda)\left[(-4-\lambda)(5-\lambda) + 18 \right]$$

$$= (-1-\lambda)\left[\lambda^2 - \lambda - 2 \right]$$

$$= (-1-\lambda)(\lambda - 2)(\lambda+1)$$

$$= -(\lambda-2)(\lambda+1)^2$$

So $\lambda = 2$, $\lambda = -1$ (mult. 2).

2 Eigenvectors:

$\lambda = 2$:
$$\begin{bmatrix} -6 & -9 & 3 \\ 0 & -3 & 0 \\ -6 & -18 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 0 \\ 2 & 6 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\xi_2 = 0$, $2\xi_1 - \xi_3 = 0$. Choose $\xi_3 = 2$; $\xi_1 = 1$. $\xi = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

$$x = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} e^{2t}$$

$\underline{\lambda=-1}$: $\begin{bmatrix} -3 & -9 & 3 \\ 0 & 0 & 0 \\ -6 & -18 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 \\ 2 & 6 & -2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$1\xi_1 + 3\xi_2 - \xi_3 = 0$. Let $\xi_3 = c_1$, $\xi_2 = c_2$. Then $\xi_1 = -3c_2 + c_1$

So $\xi = \begin{bmatrix} c_1 - 3c_2 \\ c_2 \\ c_1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$.

This eigenvalue has 2 linearly independent eigenvectors, giving 2 independent

<u>solns</u>:

$$x = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{-t}, \quad x = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} e^{-t}$$

(3) Gen Soln:

$$\boxed{x = c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} e^{-t}}$$

3. Fourier series for $f(x)=x^2/2$, $-2 \le x \le 2$ and $f(x+4)=f(x)$.

Soln: Note that f has period 4, so $2L=4$; $L=2$.

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2}\int_{-2}^{2} \frac{x^2}{2}\,dx = \frac{1}{2}\left(\frac{1}{6}x^3\right)\Big|_{-2}^{2} = \frac{1}{12}\left(8-(-2)^3\right) = \frac{1}{12}(2\cdot 8) = \frac{4}{3}$$

For $n \ge 1$:

~~a_n~~

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

$$= \frac{1}{2}\int_{-2}^{2} \frac{x^2}{2}\cos\left(\frac{n\pi}{2}x\right)dx$$

$$= \frac{1}{2}\left(2\int_{0}^{2} \frac{x^2}{2}\cos\left(\frac{n\pi}{2}x\right)dx\right) \quad \left[\text{since integrand is even}\right]$$

u		dv
$\frac{x^2}{2}$	$+$	$\cos\left(\frac{n\pi}{2}x\right)$
x	$-$	$\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}x\right)$
1	$+$	$-\left(\frac{2}{n\pi}\right)^2\cos\left(\frac{n\pi}{2}x\right)$
0		$-\left(\frac{2}{n\pi}\right)^3\sin\left(\frac{n\pi}{2}x\right)$

$$= \left(\frac{x^2}{2}\cdot\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}x\right) + x\left(\frac{2}{n\pi}\right)^2\cos\left(\frac{n\pi}{2}x\right) - \left(\frac{2}{n\pi}\right)^3\sin\left(\frac{n\pi}{2}x\right)\right)\Big|_0^2$$

$$= \left(\frac{4}{n\pi}\cancelto{0}{\sin(n\pi)} + 2\left(\frac{2}{n\pi}\right)^2\cos(n\pi) - \left(\frac{2}{n\pi}\right)^3\cancelto{0}{\sin(n\pi)}\right)$$

$$-\left(0 + 0 - \cancelto{0}{\left(\frac{2}{n\pi}\right)^3\sin(0)}\right)$$

$$= \frac{8}{(n\pi)^2}\cos(n\pi) = \frac{8(-1)^n}{(n\pi)^2}.$$

For $n \geq 1$:

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)}_{\text{even}}\ \underbrace{\sin\left(\frac{n\pi}{L}x\right)}_{\text{odd}}\,dx$$

$\Rightarrow$ since the integrand is a odd function, $b_n = 0$.
(Or, b_n can be obtained by integrating by parts.)

So the Fourier series is

$$\frac{x^2}{2} = \frac{a_0}{2} + \sum_{n\geq 1} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)$$

$$= \boxed{\frac{2}{3} + \sum_{n\geq 1} \frac{8(-1)^n}{(n\pi)^2}\cos\left(\frac{n\pi}{2}x\right)}$$

Also OK: $\dfrac{x^2}{2} = \boxed{\dfrac{2}{3} + \displaystyle\sum_{n\geq 1} \frac{8\cos(n\pi)}{(n\pi)^2}\cos\left(\frac{n\pi}{2}x\right)}$