1. Find all eigenvector/eigenvalue pairs for the following matrices.

(a) $\begin{bmatrix} 7 & 8 \\ -4 & -5 \end{bmatrix}$

$\begin{vmatrix} 7-\lambda & 8 \\ -4 & -5-\lambda \end{vmatrix} = 0$

$(7-\lambda)(-5-\lambda) - (-4)(8) = 0$

$\lambda^2 - 2\lambda - 35 + 32 = 0$

$\lambda^2 - 2\lambda - 3 = 0$

$(\lambda - 3)(\lambda + 1) = 0$

$\lambda = 3$: $\begin{bmatrix} 4 & 8 \\ -4 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

$\xi_2 = c$, $\xi_1 = -2c$ $\Rightarrow$ $\boxed{c\begin{bmatrix} -2 \\ 1 \end{bmatrix}}$

$\lambda = -1$: $\begin{bmatrix} 8 & 8 \\ -4 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$\xi_2 = c$, $\xi_1 = -c$ $\quad \boxed{c\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$

(b) $\begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix}$

$\begin{vmatrix} 4-\lambda & -3 \\ 6 & -2-\lambda \end{vmatrix} = 0$

$(4-\lambda)(-2-\lambda) + 18 = 0$

$\lambda^2 - 2\lambda + 10 = 0$

$\lambda = \frac{2 \pm \sqrt{4 - 4(10)}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$

$\lambda = 1+3i$:

$\begin{vmatrix} 3-3i & -3 \\ 6 & -3-3i \end{vmatrix} \xrightarrow{\cdot R_1(3+3i)} \begin{vmatrix} 18 & -9-9i \\ 6 & -3-3i \end{vmatrix}$

$\to \begin{vmatrix} 2 & -1-i \\ 2 & -1-i \end{vmatrix} \to \begin{bmatrix} 2 & -1-i \\ 0 & 0 \end{bmatrix}$

$\xi_2 = c$

$2\xi_1 + (-1-i)c = 0$

$\xi_1 = \left(\frac{1}{2} + \frac{1}{2}i\right)c$

$\Rightarrow \boxed{\xi = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}i \\ 1 \end{bmatrix} c}$

$\lambda = 1-3i$:

Take conj.:

$\xi = \overline{c\begin{bmatrix} \frac{1}{2} + \frac{1}{2}i \\ 1 \end{bmatrix}}$

$= \boxed{c\begin{bmatrix} \frac{1}{2} - \frac{1}{2}i \\ 1 \end{bmatrix}}$

(c) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ -4 & 4 & 3 \end{bmatrix}$

$\begin{vmatrix} 1-\lambda & 2 & 0 \\ 0 & -1-\lambda & 0 \\ -4 & 4 & 3-\lambda \end{vmatrix} = 0$

$\left[(1-\lambda)(-1-\lambda)(3-\lambda) + 0 + 0\right]$

$- \left[0 + 0 + 0\right] = 0$

$\lambda = 1, -1, 3$

$\lambda = 1$:

$\begin{bmatrix} 0 & 2 & 0 \\ 0 & -2 & 0 \\ -4 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\xi_3 = c$

$\xi_2 = 0$

$\xi_1 = \frac{c}{2}$

$\xi = c\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} = \boxed{c\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}$

$\lambda = -1$:

$\begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 0 \\ -4 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 8 & 4 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$

$\xi_3 = c$

$\xi_2 = -\frac{1}{2}c$

$\xi_1 = \frac{1}{2}c$

$\Rightarrow \xi = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} c = \boxed{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} c}$

2. A 2×2 system with real values.

   (a) Find the general solution to

$$\begin{aligned} x_1' &= -7x_1 + 10x_2 \\ x_2' &= -5x_1 + 8x_2 \end{aligned}.$$

   (b) Draw a phase portrait for the system above.

   (c) Find the solution with initial conditions $x_1(0) = 1$, $x_2(0) = -1$.

See next page.

1(c), cont

②

$\lambda = 3$:

$$\begin{bmatrix} -2 & 2 & 0 \\ 0 & -4 & 0 \\ -4 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\xi_3 = c$

$\xi_2 = 0$, $\xi_1 = 0$.

$$\xi = \boxed{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} c}$$

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2. (a) $x' = \begin{bmatrix} -7 & 10 \\ -5 & 8 \end{bmatrix} x$

① $\begin{vmatrix} -7-\lambda & 10 \\ -5 & 8-\lambda \end{vmatrix} = 0$

$(-7-\lambda)(8-\lambda) + 50 = 0$

$\lambda^2 - \lambda - 6 = 0$

$(\lambda - 3)(\lambda + 2) = 0$

$\lambda = -2, 3$

② $\lambda = -2$

$$\begin{bmatrix} -5 & 10 \\ -5 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$\xi = c\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda = 3$: $\begin{bmatrix} -10 & 10 \\ -5 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$\xi = c\begin{bmatrix} +1 \\ 1 \end{bmatrix}$.

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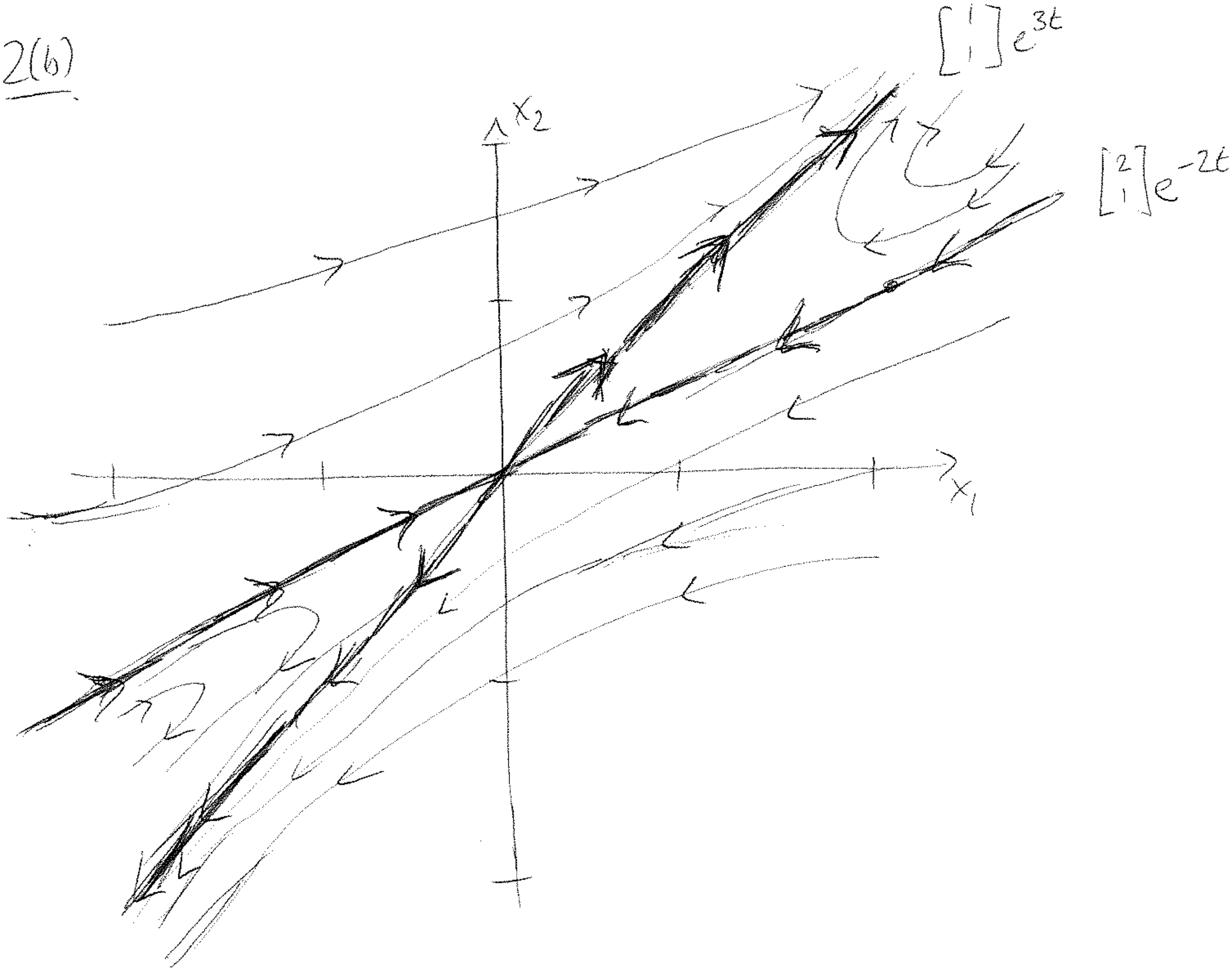
③ Solns are

$x = c\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-2t}$, $x = c\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$.

④ General solution:

$$x = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$$

2(b)



2(c) Impose $X(0) = \begin{bmatrix}1\\-1\end{bmatrix}$:

$$\begin{bmatrix}1\\-1\end{bmatrix} = c_1\begin{bmatrix}2\\1\end{bmatrix}e^{-2\cdot 0} + c_2\begin{bmatrix}1\\1\end{bmatrix}e^{3\cdot 0}$$

$$\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}2c_1+c_2\\c_1+c_2\end{bmatrix}$$

$$\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$$

$$\left[\begin{array}{cc|c}2&1&1\\1&1&-1\end{array}\right]$$

$$\sim \left[\begin{array}{cc|c}1&1&-1\\2&1&1\end{array}\right]$$

$$\sim \left[\begin{array}{cc|c}1&1&-1\\0&-1&3\end{array}\right]$$

$$\sim \left[\begin{array}{cc|c}1&0&2\\0&1&-3\end{array}\right]$$

$c_1 = 2,\ c_2 = -3$

Soln:

$$X = 2\begin{bmatrix}2\\1\end{bmatrix}e^{-2t} - 3\begin{bmatrix}1\\1\end{bmatrix}e^{3t}$$

$$= \boxed{\begin{bmatrix}4e^{-2t} - 3e^{3t}\\2e^{-2t} - 3e^{3t}\end{bmatrix}}$$