Solutions

1. Compute the following.

(a) $\mathcal{L}\{f(t)\}$ where $f(t) = \begin{cases} 0 & \text{if } t < 2 \\ t^2 & \text{if } 2 \le t \end{cases}$

$$f(t) = u_2(t)\, t^2$$
$$= u_2(t)\left[(t-2)^2 + 4t - 4\right]$$
$$= u_2(t)\left[(t-2)^2 + 4(t-2) + 4\right]$$
$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u_2(t)\left[(t-2)^2 + 4(t-2) + 4\right]\}$$
$$= e^{-2s}\,\mathcal{L}\{t^2 + 4t + 4\}$$
$$= \boxed{e^{-2s}\cdot\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right)}$$

(b) $\mathcal{L}^{-1}\left\{\frac{1-e^{-3s}}{s^2+3s+10}\right\}$

Let $H(s) = \dfrac{1}{s^2+3s+10}$.

$$\mathcal{L}^{-1}\{(1-e^{-3s})H(s)\}$$
$$= \mathcal{L}^{-1}\{H(s)\} - \mathcal{L}^{-1}\{e^{-3s}H(s)\}$$
$$= h(t) - u_3(t)\cdot h(t-3)$$

$$\mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+\frac{3}{2})^2 + 10 - \frac{9}{4}}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+\frac{3}{2})^2 + \frac{31}{4}}\right\}$$
$$= e^{-(3/2)t}\cdot\mathcal{L}^{-1}\left\{\frac{1}{s^2+\frac{31}{4}}\right\} = e^{-(\frac{3}{2}t)}\cdot\sqrt{\frac{4}{31}}\,\mathcal{L}^{-1}\left\{\frac{\sqrt{31/4}}{s^2+\frac{31}{4}}\right\}$$
$$= \frac{2}{\sqrt{31}}e^{-(3/2)t}\cdot\sin\left(\frac{\sqrt{31}}{2}t\right)$$

So $\mathcal{L}^{-1}\left\{\frac{1-e^{-3s}}{s^2+3s+10}\right\} = \boxed{\frac{2}{\sqrt{31}}e^{-\frac{3}{2}t}\sin\left(\frac{\sqrt{31}}{2}t\right) - u_3(t)\cdot\frac{2}{\sqrt{31}}e^{-\frac{3}{2}(t-3)}\sin\left(\frac{\sqrt{31}}{2}(t-3)\right)}$

(c) [7.2.19] Compute the inverse of $\begin{pmatrix} 1 & -1 & 2 & 0 \\ -1 & 2 & -4 & 2 \\ 1 & 0 & 1 & 3 \\ -2 & 2 & 0 & -1 \end{pmatrix}$.

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & -4 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ -2 & 2 & 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 3 & -1 & 0 & 1 & 0 \\ 0 & 0 & 4 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 4 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4 & -3 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & -5 & +10 & 4 & -4 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4 & -3 & -1 & +2 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & -4/5 & +4/5 & -\frac{1}{5} \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 6 & \frac{13}{5} & -\frac{8}{5} & \frac{2}{5} \\ 0 & 1 & 0 & 0 & 5 & \frac{11}{5} & -\frac{6}{5} & \frac{4}{5} \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 1 & -2 & -\frac{4}{5} & \frac{4}{5} & -\frac{1}{5} \end{array}\right]$$

So inverse is $\dfrac{1}{5}\begin{bmatrix} 30 & 13 & -8 & 2 \\ 25 & 11 & -6 & 4 \\ 0 & -1 & 1 & 1 \\ -10 & -4 & 4 & -1 \end{bmatrix}$

2. [6.4.13] Solve $y^{(4)} + 5y'' + 4y = 1 - u_\pi(t)$ with $y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0$.

$$\mathcal{L}\{y^{(4)}\} + 5\mathcal{L}\{y^{(2)}\} + 4\mathcal{L}\{y\} = \mathcal{L}\{1 - u_\pi(t)\}$$

$$\left(s^4 Y - \cancel{s^3 y(0) - s^2 y'(0) - s y''(0) - y^{(3)}(0)}\right) + 5\left(s^2 Y - \cancel{s y(0) - y'(0)}\right) + 4Y = \mathcal{L}\{1 - u_\pi(t)\}$$

$$(s^4 + 5s^2 + 4)Y = \mathcal{L}\{1\} - \mathcal{L}\{u_\pi(t)\}$$

$$(s^4 + 5s^2 + 4)Y = \frac{1}{s} - \frac{e^{-\pi s}}{s}$$

$$Y = \frac{1 - e^{-\pi s}}{s(s^4 + 5s^2 + 4)} = \frac{1 - e^{-\pi s}}{s(s^2+4)(s^2+1)} = H(s)\left(1 - e^{-\pi s}\right), \text{ where } H(s) = \frac{1}{s(s^2+4)(s^2+1)}$$

$$y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\{H(s) - e^{-\pi s} H(s)\} = h(t) - u_\pi(t) h(t - \pi)$$

$$h(t) = \mathcal{L}^{-1}\{H(t)\}$$

$$\frac{1}{s(s^2+4)(s^2+1)} = \frac{A}{s} + \frac{Bs + C}{s^2+4} + \frac{Ds + E}{s^2+1}$$

$$A(s^2+4)(s^2+1) + (Bs + C)s(s^2+1) + (Ds + E)s(s^2+4) = 1$$

$\underline{s = 0}$: $A \cdot 4 = 1$; $A = \frac{1}{4}$

$\underline{s = i}$: $0 + (Bi + C)i(\cancel{i^2 + 1}) + (Di + E)i(i^2 + 4) = 1$

$$\left.\begin{aligned} (Di + E)i \cdot 3 &= 1 \\ -3D + 3Ei &= 1 \end{aligned}\right\} \quad \left.\begin{aligned} -3D &= 1 \\ 3E &= 0. \end{aligned}\right\} \quad \begin{aligned} D &= -\frac{1}{3} \\ E &= 0. \end{aligned}$$

$\boxed{\text{Cont. on p. 4}}$

$s = \underline{2i}$:

$$\cancel{A(0)(s^2+1)} + (B(2i)+C)(2i)((2i)^2+1) + \cdots = 1$$

$$(C + 2iB)(2i)(-4+1) = 1$$

$$(C + 2iB)(-6i) = 1$$

$$12B - 6Ci = 1$$

$$\left.\begin{aligned} 12B &= 1 \\ -6C &= 0 \end{aligned}\right\} \quad \begin{aligned} B &= \tfrac{1}{12} \\ C &= 0. \end{aligned}$$

$$\frac{1}{s(s^2+4)(s^2+1)} = \frac{1/4}{s} + \frac{\frac{1}{12}s + 0}{s^2+4} + \frac{-\frac{1}{3}s + 0}{s^2+1}$$

$$= \frac{1}{4}\cdot\frac{1}{s} + \frac{1}{12}\cdot\frac{s}{s^2+2^2} - \frac{1}{3}\,\frac{s}{s^2+1}$$

$$\boxed{\begin{aligned} h(t) &= \mathcal{L}^{-1}\{H(s)\} = \tfrac{1}{4}\cdot 1 + \tfrac{1}{12}\cos(2t) - \tfrac{1}{3}\cos(t) \\ y(t) &= h(t) - u_\pi(t)\, h(t-\pi). \end{aligned}}$$