Solutions

1. Compute the following.

(a) $\mathcal{L}\{2t+e^{3t}\cosh 5t\}$ $= 2\mathcal{L}\{t\} + \left(\mathcal{L}\{\cosh 5t\}\right)\Big|_{s-3}$

$$\begin{tikzcd} f(t) \arrow[r, "\mathcal{L}"] \arrow[d, "\text{mult by exp.}"'] & F(s) \arrow[d, "\text{shift}"] \\ e^{at}f(t) \arrow[r, "\mathcal{L}"'] & F(s-a) \end{tikzcd}$$

$$= 2\cdot\frac{1}{s^2} + \left(\frac{s}{s^2-25}\right)\Big|_{s-3}$$

$$= \boxed{\frac{2}{s^2} + \frac{s-3}{(s-3)^2-25}}$$

(b) $\mathcal{L}\{f(t)\}$ where $f(t) = \begin{cases} 2 & \text{if } t<6 \\ te^t & \text{if } t \geq 6 \end{cases}$

$$f(t) = 2(1-u_6(t)) + te^t u_6(t) = 2 + (te^t-2)u_6(t) = 2 + (e^6 t e^{t-6} - 2)u_6(t)$$

$$= 2 + \left(e^6((t-6)+6)e^{t-6} - 2\right)u_6(t).$$

$$\begin{tikzcd} f(t) \arrow[r, "\mathcal{L}"] \arrow[d] & F(s) \arrow[d] \\ u_c(t)f(t-c) \arrow[r, "\mathcal{L}"'] & e^{-sc}F(s) \end{tikzcd}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{2\} + e^{-6s}\mathcal{L}\{e^6(t+6)e^t - 2\}$$

$$= \frac{2}{s} + e^{-6s}\left(\mathcal{L}\{e^6(t+6)\}\Big|_{s-1} - \frac{2}{s}\right)$$

$$= \frac{2}{s} + e^{-6s}\left(\left(\frac{e^6}{s^2} + \frac{6e^6}{s}\right)\Big|_{s-1} - \frac{2}{s}\right) \cancel{+\frac{2}{s}+e^{-6s}}$$

$$= \boxed{\frac{2}{s} + e^{-6s}\left(\frac{e^6}{(s-1)^2} + \frac{6e^6}{s-1} - \frac{2}{s}\right)}$$

This one was more complex than intended ...

(c) $\mathcal{L}^{-1}\left\{\frac{5}{s} - \frac{s}{s^2+9}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{5}{s} - \frac{s}{s^2+9}\right\} = 5\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} = 5\cdot 1 - \cos(3t)$$

$$= \boxed{5 - \cos(3t)}$$

(d) $\mathcal{L}^{-1}\left\{\frac{2s-3}{s^2+2s+10}\right\}$

$$\frac{2s-3}{s^2+2s+10} = \frac{2s-3}{(s+1)^2+9} = \frac{2(s+1)-5}{(s+1)^2+9}.$$

$$\mathcal{L}^{-1}\left\{\frac{2(s+1)-5}{(s+1)^2+9}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{2s-5}{s^2+3^2}\right\} = e^{-t}\left(2\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} - \frac{5}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}\right)$$

$$= \boxed{e^{-t}\left(2\cos(3t) - \frac{5}{3}\sin(3t)\right)}$$

2. [6.2.14] Solve $y'' - 4y' + 4y = 0$ with $y(0) = 1$ and $y'(0) = 1$.

$$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$(s^2Y - sy(0) - y'(0)) - 4(sY - y(0)) + 4Y = 0$$

$$(s^2Y - s - 1) - 4(sY - 1) + 4Y = 0$$

$$s^2Y - s - 1 - 4sY + 4 + 4Y = 0$$

$$(s^2 - 4s + 4)Y = s - 3$$

$$Y = \frac{s-3}{s^2-4s+4} = \frac{s-3}{(s-2)^2} = \frac{(s-2)-1}{(s-2)^2}$$

$$y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{(s-2)-1}{(s-2)^2}\right\} = e^{2t}\mathcal{L}^{-1}\left\{\frac{s-1}{s^2}\right\} = e^{2t}\left(\mathcal{L}^{-1}\{\tfrac{1}{s}\} - \mathcal{L}\{\tfrac{1}{s^2}\}\right)$$

$$= \boxed{e^{2t}(1-t)}$$

3. [6.2.21] Solve $y'' - 2y' + 2y = cost$ with $y(0) = 1$ and $y'(0) = 0$.

①

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\cos t\}$$

$$(s^2Y - sy(0) - y'(0)) - 2(sY - y(0)) + 2Y = \frac{s}{s^2+1}$$

$$(s^2Y - s) - 2(sY - 1) + 2Y = \frac{s}{s^2+1}$$

$$(s^2 - 2s + 2)Y - s + 2 = \frac{s}{s^2+1}$$

$$(s^2-2s+2)Y = \frac{s}{s^2+1} + s - 2$$

$$Y = \frac{s}{(s^2+1)(s^2-2s+2)} + \frac{s-2}{s^2-2s+2}$$

$$\frac{s}{(s^2+1)(s^2-2s+2)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2-2s+2}$$

$$(As+B)(s^2-2s+2)+(Cs+D)(s^2+1)=s \qquad (*)$$

Set $s=i$:

$$(Ai+B)(i^2-2i+2)+(Ci+D)(\cancelto{0}{i^2+1})=i$$

$$(Ai+B)(1-2i)=i$$

$$(B+2A)+(A-2B)i=i$$

Real Parts: $[2A+B=0]\cdot 2$

Imag. Parts: $A-2B=1$

$$5A+0B=1$$

$$A=\tfrac{1}{5}\,;\quad B=-2A=-\tfrac{2}{5}$$

~~Aaakt2~~

Expanded, $(*)$ becomes

$$(A+C)s^3+(-2A+B+D)s^2+(2A-2B+C)s+(2B+D)s^0=s$$

s^3: $A+C=0\,;\ C=-A=-\tfrac{1}{5}$

s^0: $2B+D=0 \quad D=-2B=\tfrac{4}{5}$.

So: $A=\tfrac{1}{5}$

$B=\tfrac{-2}{5}$

$C=-\tfrac{1}{5}$

$D=\tfrac{4}{5}$

$$\text{So:}\quad Y = \frac{\frac{1}{5}s - \frac{2}{5}}{s^2+1} + \frac{-\frac{1}{5}s + \frac{4}{5}}{s^2-2s+2} + \frac{s-2}{s^2-2s+2}$$

$$= \frac{1}{5}\,\frac{s}{s^2+1} - \frac{2}{5}\,\frac{1}{s^2+1} + \frac{\frac{4}{5}s - \frac{6}{5}}{s^2-2s+2}$$

$$= \frac{1}{5}\,\frac{s}{s^2+1} - \frac{2}{5}\,\frac{1}{s^2+1} + \frac{1}{5}\,\frac{4s-6}{(s-1)^2+1}$$

$$= \frac{1}{5}\,\frac{s}{s^2+1} - \frac{2}{5}\cdot\frac{1}{s^2+1} + \frac{4}{5}\cdot\frac{4(s-1)-2}{(s-1)^2+1}$$

$$= \frac{1}{5}\cdot\frac{s}{s^2+1} - \frac{2}{5}\cdot\frac{1}{s^2+1} + \frac{4}{5}\cdot\frac{s-1}{(s-1)^2+1} - \frac{2}{5}\,\frac{1}{(s-1)^2+1}$$

$$y(t) = \mathcal{L}^{-1}\{Y\} = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \frac{4}{5}\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\} - \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}$$

$$= \frac{1}{5}\cdot\cos(t) - \frac{2}{5}\sin(t) + \frac{4}{5}e^t\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{2}{5}e^t\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \frac{1}{5}\cos(t) - \frac{2}{5}\sin(t) + \frac{4}{5}e^t\cos(t) - \frac{2}{5}e^t\sin(t)$$