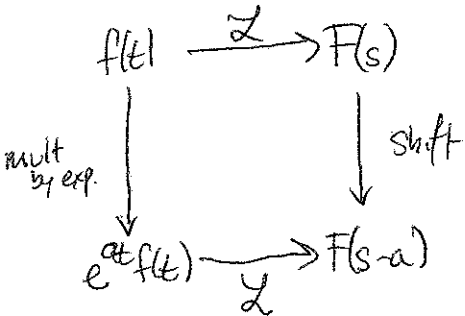


Solutions

1. Compute the following.

(a) $\mathcal{L}\{2t + e^{3t} \cosh 5t\} = 2\mathcal{L}\{t\} + \left(\mathcal{L}\{\cosh 5t\}\right)\Big|_{s-3}$



$$= 2 \cdot \frac{1}{s^2} + \left(\frac{s}{s^2 - 25}\right)\Big|_{s-3}$$

$$= \boxed{\frac{2}{s^2} + \frac{s-3}{(s-3)^2 - 25}}$$

(b) $\mathcal{L}\{f(t)\}$ where $f(t) = \begin{cases} 2 & \text{if } t < 6 \\ te^t & \text{if } t \geq 6 \end{cases}$

$$f(t) = 2(1 - u_6(t)) + te^t u_6(t) = 2 + (te^t - 2)u_6(t) = 2 + (e^6 t e^{t-6} - 2)u_6(t)$$

$$= 2 + (e^6((t-6)+6)e^{t-6} - 2)u_6(t)$$

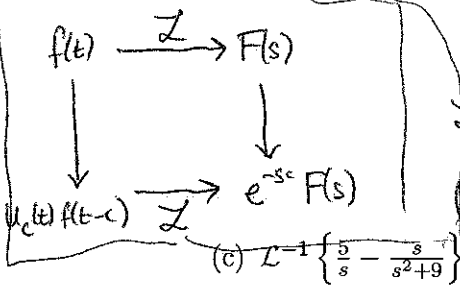
$$\mathcal{L}\{f(t)\} = \mathcal{L}\{2\} + e^{-6s} \mathcal{L}\{e^6(t+6)e^t - 2\}$$

$$= \frac{2}{s} + e^{-6s} \left(\mathcal{L}\{e^6(t+6)\}\Big|_{s-1} - \frac{2}{s} \right)$$

This one was more complex than I intended...

$$= \frac{2}{s} + e^{-6s} \left(\left(\frac{e^6}{s^2} + \frac{6e^6}{s}\right)\Big|_{s-1} - \frac{2}{s} \right) \left(\frac{2}{s} + e^{-6s} \right)$$

$$= \boxed{\frac{2}{s} + e^{-6s} \left(\frac{e^6}{(s-1)^2} + \frac{6e^6}{s-1} - \frac{2}{s} \right)}$$



(c) $\mathcal{L}^{-1}\left\{\frac{5}{s} - \frac{s}{s^2+9}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{5}{s} - \frac{s}{s^2+9}\right\} = 5\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} = 5 \cdot 1 - \cos(3t)$$

$$= \boxed{5 - \cos(3t)}$$

(d) $\mathcal{L}^{-1}\left\{\frac{2s-3}{s^2+2s+10}\right\}$

$$\frac{2s-3}{s^2+2s+10} = \frac{2s-3}{(s+1)^2+9} = \frac{2(s+1)-5}{(s+1)^2+9}$$

$$\mathcal{L}^{-1}\left\{\frac{2(s+1)-5}{(s+1)^2+9}\right\} = e^{-t} \mathcal{L}^{-1}\left\{\frac{2s-5}{s^2+3^2}\right\} = e^{-t} \left(2\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} - \frac{5}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\} \right)$$

$$= \boxed{e^{-t} \left(2\cos(3t) - \frac{5}{3}\sin(3t) \right)}$$

2. [6.2.14] Solve $y'' - 4y' + 4y = 0$ with $y(0) = 1$ and $y'(0) = 1$.

$$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$(s^2Y - sy(0) - y'(0)) - 4(sY - y(0)) + 4Y = 0$$

$$(s^2Y - s - 1) - 4(sY - 1) + 4Y = 0$$

$$s^2Y - s - 1 - 4sY + 4 + 4Y = 0$$

$$(s^2 - 4s + 4)Y = s - 3$$

$$Y = \frac{s-3}{s^2-4s+4} = \frac{s-3}{(s-2)^2} = \frac{(s-2)-1}{(s-2)^2}$$

$$y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{(s-2)-1}{(s-2)^2}\right\} = e^{2t} \mathcal{L}^{-1}\left\{\frac{s-1}{s^2}\right\} = e^{2t} (\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\})$$

$$= \boxed{e^{2t}(1-t)}$$

3. [6.2.21] Solve $y'' - 2y' + 2y = \cos t$ with $y(0) = 1$ and $y'(0) = 0$.

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\cos t\}$$

$$(s^2Y - sy(0) - y'(0)) - 2(sY - y(0)) + 2Y = \frac{s}{s^2+1}$$

$$(s^2Y - s) - 2(sY - 1) + 2Y = \frac{s}{s^2+1}$$

$$(s^2 - 2s + 2)Y - s + 2 = \frac{s}{s^2+1}$$

$$(s^2 - 2s + 2)Y = \frac{s}{s^2+1} + s - 2$$

$$Y = \frac{s}{(s^2+1)(s^2-2s+2)} + \frac{s-2}{s^2-2s+2}$$

$$\frac{s}{(s^2+1)(s^2-2s+2)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2-2s+2}$$

(3)

$$(As+B)(s^2-2s+2) + (Cs+D)(s^2+1) = s \quad (*)$$

Set $s=i$:

$$(Ai+B)(i^2-2i+2) + (Ci+D)(\cancel{i^2+1}) = i$$

$$(Ai+B)(1-2i) = i$$

$$(B+2A) + (A-2B)i = i$$

Real Parts: $[2A+B=0] \cdot 2$

Imag. Parts: $A-2B=1$

$$5A + 0B = 1$$

$$A = \frac{1}{5}; B = -2A = -\frac{2}{5}$$

~~Ans: 2~~

Expanded, (*) becomes

$$(A+C)s^3 + (-2A+B+D)s^2 + (2A-2B+C)s + (2B+D)s^0 = s$$

s^3 : $A+C=0; C=-A=-\frac{1}{5}$

s^1 : $A = \frac{1}{5}$

s^0 : $2B+D=0 \quad D = -2B = \frac{4}{5}$

$B = -\frac{2}{5}$

$C = -\frac{1}{5}$

$D = \frac{4}{5}$

(4)

So: $Y = \frac{\frac{1}{5}s - \frac{2}{5}}{s^2+1} + \frac{-\frac{1}{5}s + \frac{4}{5}}{s^2 - 2s + 2} + \frac{s-2}{s^2-2s+2}$

$$= \frac{1}{5} \frac{s}{s^2+1} - \frac{2}{5} \frac{1}{s^2+1} + \frac{\frac{4}{5}s - \frac{6}{5}}{s^2 - 2s + 2}$$

$$= \frac{1}{5} \frac{s}{s^2+1} - \frac{2}{5} \frac{1}{s^2+1} + \frac{1}{5} \frac{4s-6}{(s-1)^2+1}$$

$$= \frac{1}{5} \frac{s}{s^2+1} - \frac{2}{5} \frac{1}{s^2+1} + \frac{4}{5} \cdot \frac{4(s-1)-2}{(s-1)^2+1}$$

$$= \frac{1}{5} \cdot \frac{s}{s^2+1} - \frac{2}{5} \cdot \frac{1}{s^2+1} + \frac{4}{5} \cdot \frac{s-1}{(s-1)^2+1} - \frac{2}{5} \frac{1}{(s-1)^2+1}$$

$$y(t) = \mathcal{L}^{-1}\{Y\} = \frac{1}{5} \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{2}{5} \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \frac{4}{5} \mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\} - \frac{2}{5} \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}$$

$$= \frac{1}{5} \cdot \cos(t) - \frac{2}{5} \sin(t) + \frac{4}{5} e^t \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{2}{5} e^t \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \frac{1}{5} \cos(t) - \frac{2}{5} \sin(t) + \frac{4}{5} e^t \cos(t) - \frac{2}{5} e^t \sin(t)$$