Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[4 parts, 8 points each]** Compute the following.

(a) $\mathcal{L}\{1+t^3+te^t\}$

$= \mathcal{L}\{1\} + \mathcal{L}\{t^3\} + \mathcal{L}\{e^t \cdot t\}$

$= \frac{1}{s} + \frac{3!}{s^4} + \left(\mathcal{L}\{t\}\right)\Big|_{s-1}$

$= \frac{1}{s} + \frac{6}{s^4} + \left(\frac{1}{s^2}\right)\Big|_{s-1}$

$= \boxed{\frac{1}{s} + \frac{6}{s^4} + \frac{1}{(s-1)^2}}$

(b) $\mathcal{L}\{2\sinh(3t) - u_5(t)\,\cancel{\cos(t)}\ t\}$

$= 2\mathcal{L}\{\sinh(3t)\} - \mathcal{L}\{u_5(t)\cos(t)\}$

$= 2 \cdot \frac{3}{s^2-9} - e^{-5s} \cdot \mathcal{L}\{t+5\}$

$= \boxed{\frac{6}{s^2-9} - e^{-5s}\left(\frac{1}{s^2} + \frac{5}{s}\right)}$

(c) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s+4)}\right\}$

$\boxed{\frac{1}{s(s+4)} = \frac{A}{s} + \frac{B}{s+4}, \quad A(s+4) + Bs = 1 \quad \Rightarrow A = \frac{1}{4},\ B = -\frac{1}{4}}$

$\Rightarrow = \mathcal{L}^{-1}\left\{\frac{1}{4}\cdot\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{4}\cdot\frac{1}{s+4}\right\}$

$= \boxed{\frac{1}{4} - \frac{1}{4}e^{-4t}}$

ALT Soln: $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2-4}\right\}$

$= e^{-2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2-4}\right\} = \boxed{\frac{e^{-2t}}{2}\sinh(2t)}$

(d) $\mathcal{L}^{-1}\left\{\frac{s}{(s+6)^7}\right\}$

$= e^{-6t}\mathcal{L}^{-1}\left\{\frac{s-6}{s^7}\right\}$

$= e^{-6t}\mathcal{L}^{-1}\left\{\frac{1}{s^6} - 6\cdot\frac{1}{s^7}\right\}$

$\cancel{e^{-6t}\left(\frac{5!}{t^5}\right)}$

$= \frac{e^{-6t}}{5!}\left(\mathcal{L}^{-1}\left\{\frac{5!}{s^6}\right\} - \mathcal{L}^{-1}\left\{\frac{6\cdot 5!}{s^7}\right\}\right)$

$= \frac{e^{-6t}}{5!}\left(t^5 - t^6\right) = \boxed{\frac{t^5 e^{-6t}}{120}(1-t)}$

2. An undamped spring/mass system satisfies the equation $y'' + y = 0$. Initially, the system starts at rest. At time $t = 0$, an external motor is switched on and imparts a constant force of 1 unit. At time $t = 3$, the motor is turned off.

(a) [**4 points**] Complete the IVP $y'' + y = g(t)$, $y(0) = 0$, $y'(0) = 0$ by expressing $g(t)$ in terms of step functions.

$$g(t) = (u_0(t) - u_3(t)) \cdot 1$$

(b) [**16 points**] Solve the IVP.

① $\mathcal{L}\{g(t)\} = \mathcal{L}\{u_0(t)\} - \mathcal{L}\{u_3(t)\}$

$$= e^{-0s} \cdot \frac{1}{s} - \frac{e^{-3s}}{s}$$

$$= \frac{1}{s}(1 - e^{-3s}).$$

② $\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$

$$s^2 Y - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} + Y = \frac{1}{s}(1 - e^{-3s})$$

$$(s^2+1)Y = \frac{1}{s}(1 - e^{-3s})$$

$$Y = \frac{1}{s(s^2+1)}(1 - e^{-3s}).$$

③ Let $H(s) = \frac{1}{s(s^2+1)}$. Then

$$y(t) = \mathcal{L}^{-1}\{H(s)(1 - e^{-3s})\}$$

$$= \mathcal{L}^{-1}\{H(s)\} - u_3(t)\left(\mathcal{L}^{-1}\{H(s)\}\right)\Big|_{t-3}$$

$y(t) = h(t) - u_3(t) \cdot h(t-3).$

④ $h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}; \quad A(s^2+1) + s(Bs+C) = 1$$

$\underline{s=0}$: $A = 1$, $\underline{s=i}$: $-B + Ci = 1$

$B = -1$, $C = 0$.

$$h(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\} = 1 - \cos(t)$$

⑤ Therefore

$$y(t) = \boxed{(1 - \cos(t)) - u_3(t)(1 - \cos(t-3))}$$

for $t > 0$.

Alt soln: $\boxed{A = 2\sin(\tfrac{3}{2})}$

(c) [**5 bonus points**] For $t \geq 3$, determine the amplitude of the oscillation exactly.

For $t > 3$, $u_3(t) = 1$ and so

$$y(t) = (1 - \cos(t)) - (1 - \cos(t-3))$$

$$= \cos(t-3) - \cos(t)$$

$$= \cos(t)\cos(-3) - \sin(t)\sin(-3) - \cos(t)$$

$$= (\cos(3) - 1)\cos(t) + \sin(3)\sin(t).$$

$= A\cos(t - \delta)$, where $A\cos(\delta) = \cos(3) - 1$

$A\sin(\delta) = \sin(3).$

$$A^2\cos^2(\delta) + A^2\sin^2(\delta) = (\cos(3) - 1)^2 + (\sin(3))^2$$

$$A^2 = \cos^2(3) - 2\cos(3) + 1 + \sin^2(3)$$

$$= 2(1 - \cos(3)). \text{ So } \boxed{A = \sqrt{2(1-\cos(3))}}.$$

3. **[14 points]** Compute the inverse of $\begin{bmatrix} -2 & 1 & 1 \\ 9 & -4 & 0 \\ -8 & 4 & 2 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} -2 & 1 & 1 & 1 & 0 & 0 \\ 9 & -4 & 0 & 0 & 1 & 0 \\ -8 & 4 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} -2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 1 \\ -8 & 4 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 1 \\ -2 & 1 & 1 & 1 & 0 & 0 \\ -8 & 4 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 1 \\ 0 & 1 & 5 & 1 & 2 & 2 \\ 0 & 4 & 18 & 0 & 8 & 9 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 1 \\ 0 & 1 & 5 & 1 & 2 & 2 \\ 0 & 0 & -2 & -4 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 1 & 2 \\ 0 & 1 & 5 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 0 & -\frac{1}{2} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 1 & 2 \\ 0 & 1 & 0 & -9 & 2 & +\frac{9}{2} \\ 0 & 0 & 1 & 2 & 0 & -\frac{1}{2} \end{array}\right]$$

Inverse is $\boxed{\begin{bmatrix} -4 & 1 & 2 \\ -9 & 2 & 9/2 \\ 2 & 0 & -\frac{1}{2} \end{bmatrix}}$

4. **[14 points]** Find the eigenvector/eigenvalue pairs for the matrix $\begin{bmatrix} 3 & 2 \\ -13 & 1 \end{bmatrix}$.

$$\begin{vmatrix} 3-\lambda & 2 \\ -13 & 1-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(1-\lambda) - (2)(-13) = 0$$

$$\lambda^2 - 4\lambda + 3 + 26 = 0$$

$$\lambda^2 - 4\lambda + 29 = 0$$

$$\lambda = \frac{4 \pm \sqrt{16 - 4(29)}}{2}$$

$$= \frac{4 \pm 2\sqrt{4-29}}{2}$$

$$= 2 \pm \sqrt{-25}$$

$$= 2 \pm 5i$$

• $\lambda = 2+5i$:

$$\begin{bmatrix} 3-(2+5i) & 2 \\ -13 & 1-(2+5i) \end{bmatrix}$$

$$\sim \begin{bmatrix} 1-5i & 2 \\ -13 & -1-5i \end{bmatrix}$$

$$\sim \begin{bmatrix} 26 & 2(1+5i) \\ -13 & -(1+5i) \end{bmatrix}$$

$$\sim \begin{bmatrix} 13 & 1+5i \\ -13 & -(1+5i) \end{bmatrix}$$

$$\sim \begin{bmatrix} 13 & 1+5i \\ 0 & 0 \end{bmatrix}$$

$$13\xi_1 + (1+5i)\xi_2 = 0$$

$$\xi = c\begin{bmatrix} 1+5i \\ -13 \end{bmatrix}$$

$\lambda = 2-5i$: Complex Conjugate

$$\xi = c\begin{bmatrix} 1-5i \\ 13 \end{bmatrix}$$

So: $\lambda = 2+5i$: $c\begin{bmatrix} 1+5i \\ -13 \end{bmatrix}$

$\lambda = 2-5i$: $c\begin{bmatrix} 1-5i \\ -13 \end{bmatrix}$

5. Differential Equation System.

(a) **[16 points]** Find the general solution to $\mathbf{x}' = \begin{bmatrix} 11 & -6 \\ 4 & -3 \end{bmatrix} \mathbf{x}$.

① $\begin{vmatrix} 11-\lambda & -6 \\ 4 & -3-\lambda \end{vmatrix} = 0$

$(11-\lambda)(-3-\lambda) - (4)(-6) = 0$

$\lambda^2 - 8\lambda - 33 + 24 = 0$

$\lambda^2 - 8\lambda - 9 = 0$

$(\lambda - 9)(\lambda + 1) = 0$

$\lambda = -1, 9$

② $\underline{\lambda = -1}$:

$\begin{vmatrix} 12 & -6 \\ 4 & -2 \end{vmatrix} \sim \begin{vmatrix} 2 & -1 \\ 2 & -1 \end{vmatrix}$

$\sim \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix}$ $\quad 2\xi_1 - \xi_2 = 0$

$\xi = c\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\quad x = c\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}$

$\underline{\lambda = 9}$ $\quad \begin{bmatrix} 2 & -6 \\ 4 & -12 \end{bmatrix} \to \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix}$

$\to \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$ $\quad \xi_1 - 3\xi_2 = 0$

$\xi = c\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\quad x = c\begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{9t}$

③ So gen soln is

~~$x = c_1 [3$~~

$$\boxed{x = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{9t}}$$

(b) **[4 points]** Draw a phase portrait for the system in (a).