Name: Solutions

Directions: Show all work. No credit for answers without work.

1. [**10 points**] Solve the IVP $2y'' - y' - 3y = 0$ with $y(0) = 1$ and $y'(0) = 1$.

$2r^2 - r - 3 = 0$

$(2r-3)(r+1) = 0$

$2r - 3 = 0$ or $r = -1$

$r = \frac{3}{2}$

$y = c_1 e^{(3/2)t} + c_2 e^{-t}$

$y' = \frac{3}{2}c_1 e^{\frac{3}{2}t} - c_2 e^{-t}$

Impose $y(0)=1$:

$1 = c_1 \cdot 1 + c_2 \cdot 1$

Impose $y'(0)=1$:

$1 = \frac{3}{2}c_1 \cdot 1 - c_2 \cdot 1$

$1 = c_1 + c_2$

$1 = \frac{3}{2}c_1 - c_2$

$2 = \frac{5}{2}c_1, \quad c_1 = \frac{4}{5}; \quad c_2 = 1 - c_1 = \frac{1}{5}$

$$\boxed{y = \frac{4}{5} e^{\frac{3}{2}t} + \frac{1}{5} e^{-t}}$$

2. Consider Euler's method to approximate the solution to $y' = y$ passing through $(0, 1)$.

   (a) [**10 points**] With step size h and starting with $(x_0, y_0) = (0, 1)$, use Euler's method to compute (x_1, y_1), (x_2, y_2), and (x_3, y_3). Hint: factor your answer for y_2 and y_3.

$x_1 = x_0 + h = 0 + h = h$

$y_1 = y_0 + h(y_0) = (1+h)y_0 = (1+h)\cdot 1$

$x_2 = x_1 + h = h + h = 2h$

$y_2 = y_1 + h \cdot y_1 = (1+h)y_1 = (1+h)(1+h) = (1+h)^2$

$x_3 = x_2 + h = 2h + h = 3h$

$y_3 = y_2 + h(y_2)$

$= (1+h)y_2 = (1+h)(1+h)^2$

$= (1+h)^3$

   (b) [**5 points**] Use part (a) to give formulas for x_n and y_n in terms of n and h.

$$\boxed{\begin{aligned} x_n &= nh \\ y_n &= (1+h)^n \end{aligned}}$$

3. **[15 points]** Find the general solution to $y'' - 10y' + 29y = 0$. Your final solution should involve real numbers only.

$r^2 - 10r + 29 = 0$

$r = \dfrac{10 \pm \sqrt{100 - 4 \cdot 29}}{2}$

$r = 5 \pm \sqrt{25 - 29}$

$= 5 \pm 2i$

$y_1 = e^{(5+2i)t}$

$= e^{5t} \cdot e^{2ti}$

$= e^{5t}(\cos(2t) + i\sin(2t))$

Extract Real & Imaginary parts to get fundamental family:

$y_1 = e^{5t}\cos(2t) \quad y_2 = e^{5t}\sin(2t)$

Combine:

$\boxed{y = c_1 e^{5t}\cos(2t) + c_2 e^{5t}\sin(2t)}$

4. **[15 points]** Find the general solution to $y^{(5)} + 2y^{(4)} - 3y^{(3)} = 0$. Your final solution should involve real numbers only.

$r^5 + 2r^4 - 3r^3 = 0$

$r^3(r^2 + 2r - 3) = 0$

$r^3(r+3)(r-1) = 0$

$r=0$ mult 3	$r=-3$ mult 1	$r=1$ mult 1
$y_1 = e^{0t} = 1$	$y_4 = e^{-3t}$	$y_5 = e^t$
$y_2 = t$		
$y_3 = t^2$		

Gen soln:

$\boxed{y = c_1 + c_2 t + c_3 t^2 + c_4 e^{-3t} + c_5 e^t}$

5. **[5 points]** Write a differential equation whose general solution is $y = c_1 + c_2e^{-2t} + c_3te^{-2t}$.

Root 0, multiplicity 1

Root -2, multiplicity 2.

Char Eqn: $(r-0)(r-(-2))^2 = 0$

$r(r+2)^2 = 0$

$r(r^2+4r+4) = 0$

$r^3+4r^2+4r = 0$

$$\boxed{y^{(3)} + 4y'' + 4y' = 0}$$

6. **[15 points]** Find the general solution to $y'' - 3y' + 2y = 4e^{-t} + t$.

① Gen soln to homogeneous Eqn:

$r^2 - 3r + 2 = 0$

$(r-2)(r-1) = 0$

$y = c_1e^{2t} + c_2e^{t}$

② Find particular soln:

e^{-t} | $\cancel{-e^{-t}}$

t | 1 | $\cancel{0}$

$Y(t) = Ae^{-t} + Bt + C$

$Y'(t) = -Ae^{-t} + B$

$Y''(t) = Ae^{-t}$

$Y'' - 3Y' + 2Y = (A + 3A + 2A)e^{-t}$

$+ (0 - 3\cdot 0 + 2B)t$

$+ (0 - 3B + 2C)1$

$= 6Ae^{-t} + 2Bt + (-3B + 2C)1$

$= 4e^{-t} + 1t + 0\cdot 1$

- $6A = 4 \Rightarrow A = \frac{4}{6} = \frac{2}{3}$
- $2B = 1 \Rightarrow B = \frac{1}{2}$
- $-3B + 2C = 0 \Rightarrow 2C = 3B \Rightarrow C = \frac{3}{2}\cdot\frac{1}{2} = \frac{3}{4}$.

$Y = \frac{2}{3}e^{-t} + \frac{1}{2}t + \frac{3}{4}$

③ Gen Soln:

$$\boxed{y = c_1e^{2t} + c_2e^{t} + \frac{2}{3}e^{-t} + \frac{1}{2}t + \frac{3}{4}}$$

Find the general soln to

7. **[15 points]** ~~Solve the IVP~~ $y'' - y = e^t$ ~~with $y(0) = 0$ and $y'(0) = 0$~~.

① Gen soln to homogeneous eqn:

$r^2 - 1 = 0$

$(r+1)(r-1) = 0$

$r = -1, r = 1.$

$y = c_1 e^{-t} + c_2 e^t$

② Find Particular Soln:

Note: Overlap between homogeneous term and RHS.

e^t | e^t $\qquad Y = Ate^t$

correct for overlap

$Y = Ate^t$

$Y' = Ae^t + Ate^t$

$Y'' = Ae^t + Ae^t + Ate^t$

$= 2Ae^t + Ate^t.$

$Y'' - Y = 2Ae^t + \cancel{Ate^t} - \cancel{Ate^t}$

$= e^t$

- $2A = 1, \ A = \frac{1}{2}.$
- $Y = \frac{1}{2}te^t.$

③ Gen soln: $\boxed{y = c_1 e^{-t} + c_2 e^t + \frac{1}{2}te^t}$

8. An object with mass m, where $m > 1$, is attached to a spring. The resulting position function u satisfies the equation $mu'' + 2u' +$ [redacted] $u = 0$.

(a) **[5 points]** Determine the quasi period as a function of mass m.

$mr^2 + 2r +$ [redacted] $= 0$

$r = \dfrac{-2 \pm \sqrt{4 - 4\text{[redacted]}m}}{2m}$

$= -\dfrac{1}{m} \pm \dfrac{\sqrt{1-m}}{m}$

$r = -\dfrac{1}{m} \pm \dfrac{\sqrt{m-1}}{m} i$

$T = \dfrac{2\pi}{\mu} = \dfrac{2\pi}{\frac{\sqrt{m-1}}{m}} = \boxed{\dfrac{2\pi m}{\sqrt{m-1}}}$

(b) **[5 points]** Determine the mass that gives the shortest possible quasi period.

Find m to minimize $T(m)$:

$\dfrac{dT}{dm} = \dfrac{d}{dm}\left(2\pi m(m-1)^{-\frac{1}{2}}\right)$

$= 2\pi\left((m-1)^{-\frac{1}{2}} - \frac{1}{2}m(m-1)^{-\frac{3}{2}}\right)$

$0 = \cancel{2\pi}\left((m-1)^{-\frac{1}{2}} - \frac{1}{2}m(m-1)^{-\frac{3}{2}}\right)$

$\frac{1}{2}m(m-1)^{-\frac{3}{2}} = (m-1)^{-\frac{1}{2}}$

$\frac{1}{2}m = (m-1)$

$\boxed{m = 2}$