Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[2 parts, 15 points each]** Find the general solution explicitly.

(a) $\dfrac{dy}{dx} = 2xy$ Seperable (also linear)

$\frac{1}{y}\,dy = 2x\,dx$ or $(y=0)$

$\int \frac{1}{y}\,dy = \int 2x\,dx$

$\ln|y| = x^2 + C$

$\boxed{y = Ce^{x^2}}$ ~~or $y=0$~~ redundant

(b) $y' + \dfrac{1}{2t-1}y = 1$

Linear

$\mu = e^{\int p(t)dt}$

$= e^{\int \frac{1}{2t-1}dt}$

$= e^{\frac{1}{2}\ln(2t-1)}$

$= \sqrt{2t-1}$

$\sqrt{2t-1}\,y' + \frac{1}{\sqrt{2t-1}}y = \sqrt{2t-1}$

$\frac{d}{dt}\left[\sqrt{2t-1}\cdot y\right] = \sqrt{2t-1}$

$\sqrt{2t-1}\cdot y = \int (2t-1)^{\frac{1}{2}}\,dt$

$(2t-1)^{\frac{1}{2}}\,y = \frac{2}{3}\cdot\frac{1}{2}(2t-1)^{3/2} + C$

$(2t-1)^{\frac{1}{2}}\,y = \frac{1}{3}(2t-1)^{3/2} + C$

$\boxed{y = \frac{1}{3}(2t-1) + \frac{C}{\sqrt{2t-1}}}$

2. **[2 parts, 15 points each]** Solve the following IVPs, explicitly if possible.

(a) $\underbrace{(2x + y\sec^2 x)}_{M} + \underbrace{(3 + \tan x)}_{N}\frac{dy}{dx} = 0$ with $y(0) = 1$

$M_y = \sec^2 x$
$N_x = \sec^2 x$ } Exact ✓

① Impose $\psi_y = N$

$\psi = \int (3 + \tan x)\,dy$

$= (3 + \tan x)y + g(x)$

② Impose $\psi_x = M$

$\frac{\partial}{\partial x}\left[(3+\tan x)y + g(x)\right] = 2x + y\sec^2 x$

$y(0 + \sec^2 x) + g'(x) = 2x + y\sec^2 x$

$g'(x) = 2x$

$g(x) = \int 2x\,dx = x^2 + C$

$\psi = (3 + \tan x)y + x^2$

$= 3y + y\tan x + x^2$

Gen soln: $3y + y\tan x + x^2 = C$

Impose $y(0) = 1$: $3(1) + \tan(0) + 0^2 = C$

$3 = C$

$3y + y\tan x + x^2 = 3$

$y(\tan x + 3) = 3 - x^2$

$\boxed{y = \frac{3 - x^2}{\tan x + 3}}$

(b) $\frac{dy}{dt} = t(y + t^2)$ with $y(0) = 0$

$\frac{dy}{dt} = ty + t^3$

$\frac{dy}{dt} - ty = t^3$ Linear.

$\mu = e^{\int -t\,dt} = e^{-t^2/2}$

$e^{-t^2/2}\frac{dy}{dt} - te^{-t^2/2}y = t^3 e^{-t^2/2}$

$\frac{d}{dt}\left[e^{-t^2/2}y\right] = t^3 e^{-t^2/2}$

$e^{-t^2/2}y = \int t^3 e^{-t^2/2}\,dt$

$e^{-t^2/2}y = 2\int -\frac{t^2}{2}e^{-t^2/2}(-t)\,dt$ $\quad w = -t^2/2$, $dw = -t\,dt$

$e^{-t^2/2}y = 2\int w e^w\,dw$ $\quad u = w$, $v = e^w$, $du = dw$, $dv = e^w dw$

$e^{-t^2/2}y = 2\left(we^w - \int e^w\,dw\right)$

$e^{-t^2/2}y = 2(we^w - e^w) + C$

$e^{-t^2/2}y = 2\left(-\frac{t^2}{2} - 1\right)e^{-t^2/2} + C$

Impose $y(0) = 0$: $0 = 2(0 - 1)\cdot 1 + C$, $C = 2$.

$y = 2\left(-\frac{t^2}{2} - 1\right) + 2e^{t^2/2}$

$\boxed{y = 2e^{t^2/2} - t^2 - 2}$

3. **[2 parts, 5 points each]** A drink chilled to 45°F is taken out of the refrigerator at time $t = 0$ and placed in a warm room. Newton's law of cooling states that an object cools (or warms) at a rate proportional to the difference between the temperature of the object and the temperature of its ambient environment; the drink has proportionality constant $k = \frac{1}{2}$ 1/(hours). Let $Q(t)$ be the temperature of the object (in °F) at time t (hours).

   (a) Suppose that the room's temperature is a constant 72°F. Write a differential equation for Q. Do not solve.

$$\boxed{\frac{dQ}{dt} = \frac{1}{2}(72 - Q)}$$

   (b) Suppose instead that the air conditioner is turned off at time $t = 0$ and the room temperature steadily rises from 72°F to 80°F over the course of 5 hours. Write a differential equation for Q, valid for $0 < t < 5$. Do not solve.

Ambient Temperature: $A(t) = mt + 72$

$$\text{Slope} = m = \frac{80-72}{5} = \frac{8}{5}$$

$$A(t) = \frac{8}{5}t + 72$$

$$\frac{dQ}{dt} = \frac{1}{2}(A(t) - Q(t))$$

$$\boxed{\frac{dQ}{dt} = \frac{1}{2}\left(\frac{8}{5}t + 72 - Q\right)}$$

4. **[10 points]** Find all real numbers α such that $y = \alpha t$ is a solution to $\dfrac{dy}{dt} = \dfrac{y+t}{y-t}$.

Impose $\dfrac{dy}{dt} = \dfrac{y+t}{y-t}$:

$$\frac{d}{dt}[\alpha t] = \frac{\alpha t + t}{\alpha t - t}$$

$$\alpha = \frac{\cancel{t}(\alpha + 1)}{\cancel{t}(\alpha - 1)}$$

$$\alpha(\alpha - 1) = \alpha + 1$$

$$\alpha^2 - \alpha = \alpha + 1$$

$$\alpha^2 - 2\alpha - 1 = 0$$

$$\alpha = \frac{2 \pm \sqrt{4 - 4(-1)}}{2}$$

$$\alpha = \frac{2 \pm 2\sqrt{1+1}}{2}$$

$$\alpha = 1 \pm \sqrt{2}.$$

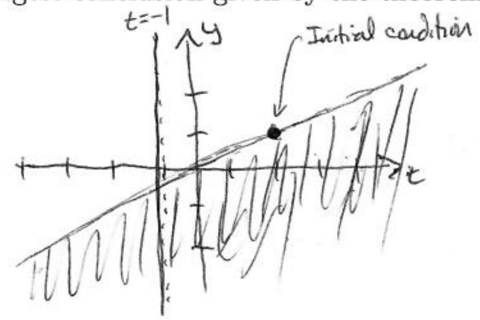
So $\boxed{\alpha = 1 - \sqrt{2} \text{ and } \alpha = 1 + \sqrt{2}}$ both lead to solutions of the form $y = (1-\sqrt{2})t$ and $y = (1+\sqrt{2})t$.

5. **[2 parts, 5 points each]** If possible, apply the Existence and Uniqueness Theorems to the following differential equations; state the strongest conclusion given by the theorems.

(a) $\dfrac{dy}{dt} = \dfrac{\sqrt{2y-t}}{t+1}$ with $y(2) = 1$

Nonlinear.

- $f = \dfrac{\sqrt{2y-t}}{t+1}$. Continuous everywhere except $t=-1$ and $2y-t<0$, or $y<\frac{t}{2}$.
- $\dfrac{\partial f}{\partial y} = \dfrac{1}{t+1}\cdot\frac{1}{2}(2y-t)^{-\frac{1}{2}}\cdot 2 = \dfrac{1}{(t+1)\sqrt{2y-t}}$.
- $\dfrac{\partial f}{\partial y}$ continuous except when $t=-1$, $2y-t\le 0$, $y\le t/2$.



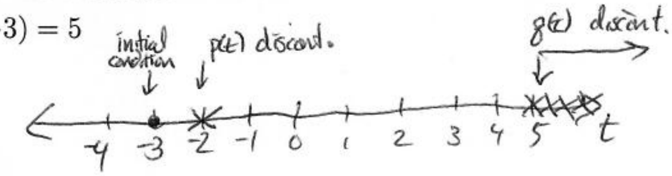
- Since there is no open rectangle containing $(2,1)$ in which f and $\frac{\partial f}{\partial y}$ are continuous,

$\boxed{\text{Thm NLF gives no conclusion.}}$

(b) $(t+2)y' = (t+2)\ln(5-t) - y$, with $y(-3) = 5$

Linear: $y' + \frac{1}{t+2}y = \ln(5-t)$

- $p(t) = \frac{1}{t+2}$ continuous except at $t=-2$.
- $g(t) = \ln(5-t)$ continuous except at $5-t\le 0$ or $t\ge 5$



- $\boxed{\text{Thm LF implies that there is a unique solution on } (-\infty, -2).}$

6. **[10 points]** Identify the equilibrium solution(s) of $y' = -(y^2 + 2y - 15)$, and classify each as stable, semistable, or unstable. Sketch the solutions (with phase line).

- $y' = -(y+5)(y-3)$

Critical pts:

$0 = -(y+5)(y-3)$

$y=-5$ or $y=3$.

$\boxed{\begin{array}{l} y=-5:\ \text{unstable} \\ y=3:\ \text{stable} \end{array}}$