Name: Solutions

1. **[3 points]** Find the general solution to $y^{(4)} + y^{(3)} - 8y'' - 12y' = 0$.

$r^4 + r^3 - 8r^2 - 12r = 0$

$r(r^3 + r^2 - 8r - 12) = 0$

candidate roots: $\pm 1, \pm 2, \pm 3, \pm 4$
$\pm 6, \pm 12$

$r = -2$ is a root

$$\begin{array}{r} r^2 - r - 6 \\ r+2 \,\big)\, \overline{r^3 + r^2 - 8r - 12} \\ -(r^3 + 2r^2) \\ \hline -r^2 - 8r \\ -(-r^2 - 2r) \\ \hline -6r - 12 \\ -6r - 12 \\ \hline 0 \end{array}$$

$r(r+2)(r^2 - r - 6) = 0$

$r(r+2)(r-3)(r+2) = 0$

$r = 0$, $r = -2$ (mult 2), $r = 3$

$y_1 = e^{0t} = 1$, $y_2 = e^{-2t}$, $y_3 = te^{-2t}$, $y_4 = e^{3t}$

So $\boxed{y = c_1 + c_2 e^{-2t} + c_3 t e^{-2t} + c_4 e^{3t}}$

2. **[3 points]** Find the general solution to $y^{(2)} - 3y' - 10y = te^{2t}$.

① $r^2 - 3r - 10 = 0$

$(r-5)(r+2) = 0$

$r = -2, 5$

$y = c_1 e^{-2t} + c_2 e^{5t}$

② te^{2t}

$e^{2t} + \cancel{2te^{2t}}$

$\cancel{2e^{2t}}$

$Y(t) = Ate^{2t} + Be^{2t}$

$Y' = Ae^{2t} + 2Ate^{2t} + 2Be^{2t}$
$= 2Ate^{2t} + (A + 2B)e^{2t}$

$Y'' = 4Ate^{2t} + (2A + 2(A + 2B))e^{2t}$
$= 4Ate^{2t} + (4A + 4B)e^{2t}$

$Y'' - 3Y' - 10y = (4A - 6A - 10A)te^{2t}$
$+ (\underbrace{4A + 4B - 3(A + 2B)}_{-10B})e^{2t}$

$(-12A)te^{2t} + (A - 12B)e^{2t} = te^{2t}$

$-12A = 1$, $A = -\frac{1}{12}$

$A - 12B = 0$, $B = \frac{A}{12} = -\frac{1}{144}$

$\boxed{y = c_1 e^{-2t} + c_2 e^{5t} - \frac{1}{12}te^{2t} - \frac{1}{144}e^{2t}}$

3. [**4 points**] Solve the IVP $y'' - 4y' + 4y = t$ with $y(0) = 0$ and $y'(0) = 1$.

(1) $r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$r = 2$, mult 2.

$y = c_1 e^{2t} + c_2 t e^{2t}$

(2). t

$|$

1

$Y(t) = At + B$

$Y' = A$

$Y'' = 0.$

$y'' - 4y' + 4y = (0 - 4 \cdot 0 + 4A)t + (0 - 4A + 4B)1$

$= 4At + (-4A + 4B)1.$

- $4A = 1$, $A = \frac{1}{4}$
- $-4A + 4B = 0$, $B = A = \frac{1}{4}$

$Y(t) = \frac{1}{4}t + \frac{1}{4}$

(3) Gen soln:

$y = c_1 e^{2t} + c_2 t e^{2t} + \frac{1}{4}t + \frac{1}{4}$

$y' = 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t e^{2t} + \frac{1}{4}.$

Impose $y(0) = 0$:

$0 = c_1 \cdot 1 + \cancel{c_2 \cdot 0 \cdot 1} + \cancel{\frac{1}{4} 0} + \frac{1}{4}$

$c_1 = -\frac{1}{4}$

Impose $y'(0) = 1$:

$1 = 2c_1 \cdot 1 + c_2 \cdot 1 + \cancel{2c_2 \cdot 0 \cdot 1} + \frac{1}{4}$

$1 = -\frac{1}{2} + c_2 + \frac{1}{4}$, $c_2 = \frac{3}{2} - \frac{1}{4} = \frac{5}{4}$

$$\boxed{y = -\frac{1}{4}e^{2t} + \frac{5}{4}te^{2t} + \frac{1}{4}t + \frac{1}{4}}$$