Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[3 points]** Find the general solution to $2y'' + 10y' + 17y = 0$. Express your answer in terms of real numbers.

Char. Eqn: $2r^2 + 10r + 17 = 0$

$$r = \frac{-10 \pm \sqrt{100 - 4(2)(17)}}{4}$$

$$r = \frac{-10 \pm \sqrt{-36}}{4}$$

$$r = \frac{-10 \pm 6i}{4}$$

$$r = -\frac{5}{2} \pm \frac{3}{2}i$$

Complex conjugate pair.

$$y = e^{(-5/2 + \frac{3}{2}i)t}$$

$$= e^{-(5/2)t}\left[e^{\frac{3}{2}t \cdot i}\right]$$

$$= e^{-(5/2)t}\left[\cos\left(\tfrac{3}{2}t\right) + i\sin\left(\tfrac{3}{2}t\right)\right]$$

$$y_1 = e^{-\frac{5}{2}t}\cos\left(\tfrac{3}{2}t\right), \quad y_2 = e^{-\frac{5}{2}t}\sin(t)$$

$$\boxed{y = c_1 e^{-\frac{5}{2}t}\cos\left(\tfrac{3}{2}t\right) + c_2 e^{-\frac{5}{2}t}\sin(t)}$$

2. **[3 points]** Solve the IVP $y'' + 5y' - 14y = 0$ with $y(0) = 2$ and $y'(0) = -1$. Express your answer in terms of real numbers.

Char Eqn:

$$r^2 + 5r - 14 = 0$$

$$(r+7)(r-2) = 0$$

$$r = -7, 2.$$

$$y_1 = e^{-7t}, \quad y_2 = e^{2t}.$$

$$y = c_1 e^{-7t} + c_2 e^{2t}.$$

$$y' = -7c_1 e^{-7t} + 2c_2 e^{2t}$$

$y(0)=2$: $2 = c_1 + c_2$

$y'(0)=-1$: $-1 = -7c_1 + 2c_2$

$$-4 = -2c_1 - 2c_2$$

$$-5 = -9c_1$$

$$c_1 = \frac{5}{9}$$

$$c_2 = 2 - c_1 = 2 - \frac{5}{9}$$

$$= \frac{13}{9}$$

$$\boxed{y = \frac{5}{9}e^{-7t} + \frac{13}{9}e^{2t}}$$

3. **[3 points]** Show that $y_1(t) = \cos 2t$ and $y_2(t) = \sin 2t$ are solutions to $y'' + 4y = 0$. Then, decide if y_1 and y_2 form a fundamental set of solutions. Show your work.

$y_1' = -2\sin(2t)$

$y_1'' = -4\cos(2t)$

$y_1'' + 4y_1 \overset{?}{=} 0$

$(-4\cos(2t)) + 4(\cos(2t)) \overset{?}{=} 0$

$0 = 0$ ✓

So y_1 is a soln.

$y_2' = 2\cos(2t)$

$y_2'' = -4\sin(2t)$

$y_2'' + 4y_2 \overset{?}{=} 0$

$(-4\sin(2t)) + 4\sin(2t) \overset{?}{=} 0$

$0 = 0$ ✓

So y_2 is a soln.

$$W = \begin{vmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{vmatrix}$$

$= 2\cos^2(2t) - (-2\sin^2(2t))$

$= 2(\cos^2(2t) + \sin^2(2t))$

$= 2$

Since $W(t) = 2 \neq 0$, y_1 and y_2 form a fundamental set of solns.

4. **[1 point]** Given that $y_1 = \cos t$, find a function y_2 so that the Wronskian $W(t)$ of y_1 and y_2 satisfies $W(t) = \cos^2 t$.

$$\begin{vmatrix} \cos t & y_2 \\ -\sin t & y_2' \end{vmatrix} = \cos^2 t$$

$\cos t\, y_2' + \sin t\, y_2 = \cos^2 t$

$y_2' + \tan t\, y_2 = \cos t$

$\mu = e^{\int \tan t \, dt}$

$= e^{-\ln|\cos t|}$

$= \frac{1}{\cos t} = \sec t.$

$\sec t \cdot y_2' + \sec t \cdot \tan t\, y_2 = \frac{1}{\cos t} \cdot \cos t$

$\frac{d}{dt}[\sec t\, y_2] = 1$

$\sec t \cdot y_2 = \int 1 \, dt$

$\sec t \cdot y_2 = t + C$. choose $C = 0$

$\frac{1}{\cos t} y_2 = t$

$\boxed{y_2 = t\cos t}$

In general, $y_2 = (t + C)\cos t$ for any const. C is a valid answer.