Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[3 parts, 1 point each]** A retirement account earns an annual interest rate of 4%, compounded continuously. The retiree has monthly expenses of \$2000, which are withdrawn continuously. Let $S(t)$ be the dollar value of the account at time t (years), with $S(0) = S_0$.

   (a) Write a differential equation for $S(t)$. (Note that t is measured in years but the given expenses are monthly.)

   \$2000/mo = \$24,000/yr.

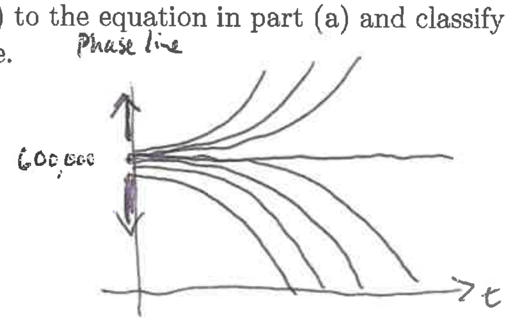
   $$\boxed{\frac{dS}{dt} = 0.04S - 24{,}000}$$

   (b) Determine the equilibrium solution(s) to the equation in part (a) and classify each solution as stable, semistable, or unstable.

   $0 = 0.04S - 24{,}000$

   $\frac{1}{25}S = 24{,}000$

   $\boxed{S = 600{,}000 \quad \text{unstable}}$



   (c) What does your answer to part (b) mean in the context of the retirement account?

   If the account initially has more than \$600,000 then it will grow forever. If it starts with less than \$600,000, it will eventually run out.

2. **[2 parts, 2 points each]** Apply the Existence and Uniqueness Theorems to the following differential equations; state the strongest conclusion given by the theorems.

   (a) $(\cos t)y' + (\sin t)y = \frac{\cos t}{2t+1}$, with $y(0) = 0$.

   $y' + \tan t \, y = \frac{\cos t}{(2t+1)(\cos t)}$

   This is a linear first-order eqn. Apply Thm LF with $p(t) = \tan(t)$, $g(t) = \frac{1}{2t+1}$.

   - $\tan(t)$ is ~~discontinuous~~ continuous everywhere except when $\cos(t) = 0$; ie. $t = \frac{\pi}{2} + k\pi$ for some $k \in \mathbb{Z}$.
   - $\frac{1}{2t+1}$ is continuous everywhere except $t = -\frac{1}{2}$.
   - $\boxed{\text{Soln exists and is unique on } (-\tfrac{1}{2}, \tfrac{\pi}{2})}$

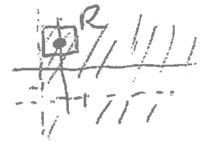
(b) $(y+1)y' = \ln(t+1)$, with $y(0) = 1$

Nonlinear; $y' = \dfrac{\ln(t+1)}{y+1}$.

Apply Thm NLF with $f(y,t) = \dfrac{\ln(t+1)}{y+1}$

• Note, $f(y,t)$ is continuous everywhere ~~except~~ with $t>-1$ and $y \neq -1$.

• $\dfrac{\partial f}{\partial y} = -\dfrac{\ln(t+1)}{(y+1)^2}$. Also continuous everywhere with $t>-1$ and $y \neq -1$.

• 

• So the solution exists and is unique in a small open neighborhood around $t=0$

3. **[3 points]** Find the general solution to $y' - \frac{1}{t}y = (\sin t)y^2$.

Bernoulli Use $v = y^{1-n} = y^{1-2} = y^{-1}$ (or $y=0$).

• $y = v^{-1}$; $\dfrac{dy}{dt} = -v^{-2}\dfrac{dv}{dt}$

• $-v^{-2}\dfrac{dv}{dt} - \dfrac{1}{t}v^{-1} = (\sin t)\,v^{-2}$

• $\dfrac{dv}{dt} + \dfrac{1}{t}v = -(\sin t)$

• $\mu = e^{\int \frac{1}{t}dt} = e^{\ln t} = t$.

• $t\dfrac{dv}{dt} + v = -t\sin t$

• $\dfrac{d}{dt}\left[tv\right] = -t\sin t$

• $tv = \int -t\sin t\,dt$ $\quad u = t \quad v = \cos t$ $\quad du = dt \quad dv = -\sin t\,dt$

• $tv = t\cos t - \int \cos t\,dt$

• $tv = t\cos t - \sin t + C$

$\dfrac{t}{y} = t\cos t - \sin t + C$

$\boxed{y = \dfrac{t}{t\cos t - \sin t + C} \text{ or } y = 0}$