Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[4 points]** Using real numbers, find the general solution to $\mathbf{x}' = \begin{bmatrix} -9 & -4 \\ 9 & 3 \end{bmatrix}\mathbf{x}$.

① Eigenvals:

$$\begin{vmatrix} -9-\lambda & -4 \\ 9 & 3-\lambda \end{vmatrix} = 0$$

$$(-9-\lambda)(3-\lambda) + 36 = 0$$

$$\lambda^2 + 6\lambda - 27 + 36 = 0$$

$$\lambda^2 + 6\lambda + 9 = 0$$

$$(\lambda+3)^2 = 0$$

$\lambda = -3$, mult 2.

② Eigenvects:

$$\begin{bmatrix} -6 & -4 \\ 9 & 6 \end{bmatrix} \sim \begin{bmatrix} -3 & -2 \\ 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix} \quad \xi = \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

$$\Rightarrow x = \begin{bmatrix} 2 \\ -3 \end{bmatrix} e^{-3t}.$$

③ Need a second soln:

$$(A - \lambda I)\eta = \xi$$

$$\left[\begin{array}{cc|c} -6 & -4 & 2 \\ 9 & 6 & -3 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 2 & -1 \\ -3 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 2 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

$3\eta_1 + 2\eta_2 = -1$

$\eta_2 = c$, $\eta_1 = (-1-2c)\frac{1}{3}$

choose $c = 1$, $\eta = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$$x = (\xi t + \eta)e^{-3t} = \left(\begin{bmatrix} 2 \\ -3 \end{bmatrix} t + \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)e^{-3t}$$

④ Gen soln:

$$\boxed{x = c_1 \begin{bmatrix} 2 \\ -3 \end{bmatrix} e^{-3t} + c_2\left(\begin{bmatrix} 2 \\ -3 \end{bmatrix} t + \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) e^{-3t}}$$

2. **[3 points]** Convert $x^{(3)} + 2x'' + 5x = 0$ to a system of first-order differential equations. Your system should be as small as possible. Do not attempt to solve.

$y_1 = x$, $y_2 = x'$, $y_3 = x''$

$y_2 = y_1'$ $\quad y_3 = y_2'$

$$x^{(3)} + 2x'' + 5x = 0$$

$$y_3' + 2y_3 + 5y_1 = 0$$

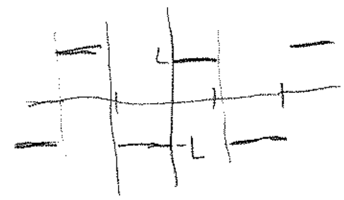
$$y_1' = y_2$$

$$y_2' = y_3$$

$$y_3' = -5y_1 - 2y_3$$

$$\boxed{y' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & 0 & -2 \end{bmatrix} y}$$

3. **[3 points]** Let $f(x) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1 \end{cases}$ with $f(x+2) = f(x)$. Find the Fourier Series for $f(x)$. Simplify your expression as much as possible.

f: 

Note: $f(x)$ is an odd function, period 2.
Set $2L = 2$, so $L = 1$.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx = \frac{1}{L}\int_{-L}^{L} \text{odd}\cdot\text{even}\,dx = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$= 2\int_0^1 \sin(n\pi x)\,dx = 2\left(-\frac{1}{n\pi}\cos(n\pi x)\right)\Big|_0^1$$

$$\cancel{= \frac{2}{n\pi}\left(-\cos(n\pi\right.}$$

$$= 2\left(-\frac{1}{n\pi}\cos(n\pi) + \frac{1}{n\pi}\cos(0)\right)$$

$$= 2\left(-\frac{1}{n\pi}(-1)^n + \frac{1}{n\pi}\right)$$

$$= \frac{2}{n\pi}\left(1-(-1)^n\right) = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{4}{n\pi} & \text{if } n \text{ is odd.} \end{cases}$$

So $$f(x) = \frac{a_0}{2} + \sum_{n\geq 1} a_n\cos\left(\frac{n\pi}{L}x\right) + b_n\sin\left(\frac{n\pi}{L}x\right)$$

$$= \boxed{\sum_{n\geq 1} \frac{2}{n\pi}\left(1-(-1)^n\right)\sin(n\pi x)}$$ also ok

$$= \boxed{\sum_{n=1,3,5,7,\ldots} \frac{4}{n\pi}\cdot\sin(n\pi x)}$$ better, simpler