Name: Solutions

Directions: Show all work. No credit for answers without work.

1. [**4 points**] Find all eigenvalue/eigenvector pairs for $\begin{bmatrix} -4 & -3 & -6 \\ 18 & 11 & 12 \\ 0 & 0 & -1 \end{bmatrix}$.

① $\begin{vmatrix} -4-\lambda & -3 & -6 \\ 18 & 11-\lambda & 12 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0$

$\left[(-4-\lambda)(11-\lambda)(-1-\lambda) + 0 + 0\right]$

$- \left[0 + 0 + (-1-\lambda)(18)(-3)\right] = 0$

$-(\lambda+1)\left[(-4-\lambda)(11-\lambda) - (18)(-3)\right] = 0$

$-(\lambda+1)\left[\lambda^2 - 7\lambda - 44 + 54\right] = 0$

$-(\lambda+1)\left[\lambda^2 - 7\lambda + 10\right] = 0$

$-(\lambda+1)(\lambda-5)(\lambda-2) = 0$

$\lambda = -1, \lambda = 2, \lambda = 5.$

② $\underline{\lambda = -1}$:

$\begin{bmatrix} -3 & -3 & -6 \\ 18 & 12 & 12 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

$\xi = \begin{bmatrix} 2c \\ -4c \\ c \end{bmatrix} = c\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$

$\underline{\lambda = 2}$ $\begin{bmatrix} -6 & -3 & -6 \\ 18 & 9 & 12 \\ 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 2 \\ 6 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 2 & 1 & 0 \\ 6 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\xi = \begin{bmatrix} -c/2 \\ c \\ 0 \end{bmatrix} = c\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} = \hat{c}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

$\underline{\lambda = 5}$ $\begin{bmatrix} -9 & -3 & -6 \\ 18 & 6 & 12 \\ 0 & 0 & -6 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & 2 \\ 3 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 3 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\xi = \begin{bmatrix} -c/3 \\ c \\ 0 \end{bmatrix} = \hat{c}\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$

③ So eigenvalue/eigenvector pairs are

$$\boxed{\begin{aligned} &\lambda = -1, \ \xi = c\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} \\ &\lambda = 2, \ \xi = c\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \\ &\lambda = 5, \ \xi = c\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} \end{aligned}}$$

(a) **[4 points]** Find the general solution to $\mathbf{x}' = \begin{bmatrix} -2 & 20 \\ -1 & 7 \end{bmatrix}\mathbf{x}$.

① $\begin{vmatrix} -2-\lambda & 20 \\ -1 & 7-\lambda \end{vmatrix} = 0$

$(-2-\lambda)(7-\lambda) - (-1)(20) = 0$

$\lambda^2 - 5\lambda - 14 + 20 = 0$

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda-2)(\lambda-3) = 0$

$\underline{\lambda = 2, 3}$.

② $\underline{\lambda = 2}$:

$\begin{bmatrix} -4 & 20 \\ -1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -5 \\ -1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix}$

$\xi = c\begin{bmatrix} 5 \\ 1 \end{bmatrix}$

$x = \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^{2t}$

$\underline{\lambda = 3}$ $\begin{bmatrix} -5 & 20 \\ -1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -4 \\ -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$ $\xi = c\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$x = \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{3t}$

③ The gen. soln is

$x = c_1\begin{bmatrix} 5 \\ 1 \end{bmatrix} e^{2t} + c_2\begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{3t}$.

(b) **[2 points]** ~~Draw a phase portrait~~ Solve the system in (a) with $x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

④ Impose $x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$:

$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_1\begin{bmatrix} 5 \\ 1 \end{bmatrix} \cdot \cancel{e^{0}} + c_2\begin{bmatrix} 4 \\ 1 \end{bmatrix}\cancel{e^{0}}$

$= \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$\left[\begin{array}{cc|c} 5 & 4 & 1 \\ 1 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 5 & 4 & 1 \end{array}\right]$

$\sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -1 & -9 \end{array}\right]$

$\sim \left[\begin{array}{cc|c} 1 & 0 & -7 \\ 0 & 1 & 9 \end{array}\right]$

$c_1 = -7, c_2 = 9$

$\boxed{x = -7\begin{bmatrix} 5 \\ 1 \end{bmatrix} e^{2t} + 9\begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{3t}}$