Name: Solutions

1. **[2 parts, 2 points each]** Compute the following.

(a) $\mathcal{L}\left\{u_{\pi/2}(t)\sin(3t)\right\}$

$\mathcal{L}\{u_{\pi/2}(t)\sin(3(t-\tfrac{\pi}{2})+\tfrac{3\pi}{2})\}$

$= e^{-(\pi/2)s}\,\mathcal{L}\{\sin(3t+\tfrac{3\pi}{2})\}$

$= e^{-(\frac{\pi}{2})s}\,\mathcal{L}\{\sin(3t)\cancelto{0}{\cos(\tfrac{3\pi}{2})} + \cos(3t)\cancelto{-1}{\sin(\tfrac{3\pi}{2})}\}$

$= e^{-(\frac{\pi}{2})s}\,\mathcal{L}\{-\cos(3t)\}$

$= \boxed{-e^{-\frac{\pi}{2}s}\cdot\frac{s}{s^2+9}}$

(b) $\mathcal{L}^{-1}\left\{\frac{e^{-6s}}{s^2+4}\right\}$ $= u_6(t)\left(\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}\right)\Big|_{t-6}$

$= u_6(t)\left(\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}\right)_{t-6}$

$= u_6(t)\left(\frac{1}{2}\sin(2t)\right)\Big|_{t-6}$

$= \boxed{\frac{u_6(t)}{2}\cdot\sin(2(t-6))}$

2. Compute the inverse of $\begin{bmatrix} -3 & 4 & -6 \\ -1 & 1 & -2 \\ -4 & 1 & -7 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} -3 & 4 & -6 & 1 & 0 & 0 \\ -1 & 1 & -2 & 0 & 1 & 0 \\ -4 & 1 & -7 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & -1 & 0 \\ -3 & 4 & -6 & 1 & 0 & 0 \\ -4 & 1 & -7 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & -3 & 0 \\ 0 & -3 & 1 & 0 & -4 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & -4 & 0 \\ 0 & 1 & 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 3 & -13 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 22 & -2 \\ 0 & 1 & 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 3 & -13 & 1 \end{array}\right]$$

So inverse is $\boxed{\begin{bmatrix} -5 & 22 & -2 \\ 1 & -3 & 0 \\ 3 & -13 & 1 \end{bmatrix}}$.

3. **[4 points]** Use the Laplace transform to solve $y''+5y'-14y=g(t)$, where $g(t)=\begin{cases}1 & \text{if } t<4\\ 0 & \text{if } t\geq 4\end{cases}$ with $y(0)=y'(0)=0$.

$g(t) = (1-u_4(t))1 + \cancel{u_4(t)0}$

$\mathcal{L}\{y''\} + 5\mathcal{L}\{y'\} - 14\mathcal{L}\{y\} = \mathcal{L}\{1-u_4(t)\}$

$(s^2Y - s\cancel{y(0)} - \cancel{y'(0)}) + 5(sY - \cancel{y(0)}) - 14Y = \frac{1}{s} - \frac{e^{-4s}}{s}$

$(s^2+5s-14)Y = \frac{1-e^{-4s}}{s}$

$Y = \frac{1-e^{-4s}}{s(s+7)(s-2)}$

① Let $H(s) = \frac{1}{s(s+7)(s-2)}$.

$y(t) = \mathcal{L}^{-1}\{Y\}$

$= \mathcal{L}^{-1}\{(1-e^{-4s})H(s)\}$

$= \mathcal{L}^{-1}\{H(s) - e^{-4s}H(s)\}$

$= h(t) - u_4(t)h(t-4)$.

② Find $h(t) = \mathcal{L}^{-1}\{H(s)\}$.

$H(s) = \frac{1}{s(s+7)(s-2)} = \frac{A}{s} + \frac{B}{s+7} + \frac{C}{s-2}$

$A(s+7)(s-2) + Bs(s-2) + Cs(s+7) = 1$

$s=2$: $C\cdot 2\cdot 9 = 1$, $C = \frac{1}{18}$

$s=-7$: $B\cdot(-7)(-9) = 1$, $B = \frac{1}{63}$

$s=0$: $A(7)(-2) = 1$, $A = -\frac{1}{14}$.

$\mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{-\frac{1}{14}\cdot\frac{1}{s} + \frac{1}{63}\cdot\frac{1}{s+7} + \frac{1}{18}\cdot\frac{1}{s-2}\right\}$

$= \underbrace{-\frac{1}{14} + \frac{1}{63}e^{-7t} + \frac{1}{18}\cdot e^{2t}}_{h(t)}$.

③

$y(t) = h(t) - u_4(t)h(t-4)$

$$\boxed{= \left(-\frac{1}{14} + \frac{1}{63}e^{-7t} + \frac{1}{18}e^{2t}\right) - u_4(t)\left[-\frac{1}{14} + \frac{1}{63}e^{-7(t-4)} + \frac{1}{18}e^{2(t-4)}\right]}$$