Name: Solutions

1. **[4 parts, 1.5 points each]** Compute the following.

   (a) $\mathcal{L}\{2 + te^{-t}\}$

$$= \mathcal{L}\{2\} + \mathcal{L}\{te^{-t}\}$$

$$= \frac{2}{s} + \left(\mathcal{L}\{t\}\right)\Big|_{s+1}$$

$$= \frac{2}{s} + \left(\frac{1}{s^2}\right)\Big|_{s+1} = \boxed{\frac{2}{s} + \frac{1}{(s+1)^2}}$$

   (b) $\mathcal{L}\{\cosh(5t) + \sin(3t)\}$

$$= \mathcal{L}\{\cosh(5t)\} + \mathcal{L}\{\sin(3t)\}$$

$$= \boxed{\frac{s}{s^2-25} + \frac{3}{s^2+9}}$$

   (c) $\mathcal{L}^{-1}\left\{\frac{1}{(s-5)^2+7}\right\} = e^{5t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+7}\right\}$

$$= e^{5t}\mathcal{L}^{-1}\left\{\frac{\sqrt{7}}{s^2+7}\right\}\cdot\frac{1}{\sqrt{7}}$$

$$= \boxed{\frac{e^{5t}}{\sqrt{7}}\cdot\sin\left(\sqrt{7}\,t\right)}$$

   (d) $\mathcal{L}^{-1}\left\{\frac{3s+1}{s^2+6s+5}\right\} = \frac{7}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$

$$= \boxed{\frac{7}{2}e^{-5t} - \frac{1}{2}e^{-t}}$$

$$\frac{3s+1}{(s+5)(s+1)} = \frac{A}{s+5} + \frac{B}{s+1}$$

$A(s+1) + B(s+5) = 3s+1$

$\underline{s=-1}$: $4B = -2$, $B = -\frac{1}{2}$

$\underline{s=-5}$: $-4A = -14$, $A = \frac{7}{2}$

<u>Note</u>: Completing the square leads to the equivalent soln $\boxed{e^{-3t}\left[3\cosh(2t) - 4\sinh(2t)\right]}$.

2. **[4 points]** Use the Laplace transform to solve $y'' - 4y' + 3y = \cancel{t+}\cos t$ with $y(0) = 1$ and $y'(0) = -1$.

$$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = \mathcal{L}\{t\} + \mathcal{L}\{\cos t\}$$

$$\left(s^2 Y - s\cdot y(0) - y'(0)\right) - 4\left(sY - y(0)\right) + 3Y = \cancel{\tfrac{1}{s^2}} + \frac{s}{s^2+1}$$

$$\left(s^2 Y - s + 1\right) - 4\left(sY - 1\right) + 3Y = \cancel{\tfrac{1}{s^2}} + \frac{s}{s^2+1}$$

$$(s^2 - 4s + 3)Y - s + 1 + 4 = \cancel{\tfrac{1}{s^2}} + \frac{s}{s^2+1}$$

$$Y = \frac{1}{(s-3)(s-1)}\left[s - 5 + \cancel{\tfrac{1}{s^2}} + \frac{s}{s^2+1}\right]$$

$$= \frac{(s-5)(s^2+1)}{(s-3)(s-1)(s^2+1)} \cancel{+ \frac{1}{s^2(s-3)(s-1)}} + \frac{s}{(s^2+1)(s-3)(s-1)}$$

$$= \frac{s^3 - 5s^2 + s - 5 + s}{(s-3)(s-1)(s^2+1)} = \frac{s^3 - 5s^2 + 2s - 5}{(s-3)(s-1)(s^2+1)} = \frac{A}{s-3} + \frac{B}{s-1} + \frac{Cs + D}{s^2+1}$$

$$A(s-1)(s^2+1) + B(s-3)(s^2+1) + (Cs + D)(s-3)(s-1) = s^3 - 5s^2 + 2s - 5$$

$s=1$: $B(-2)(2) = -7$, $B = \frac{7}{4}$

$s=3$: $A(2)(10) = 27 - 45 + 6 - 5$

$= -17$, $A = -\frac{17}{20}$

$s=i$: $(Ci + D)(i-3)(i-1) = -i + 5 + 2i - 5$

$(Ci + D)(i^2 - 4i + 3) = i$

$(Ci + D)(-4i + 2) = i$

$-4Ci^2 - 4Di + 2Ci + 2D = i$

$4C + 2D + (2C - 4D)i = i$

$2(C + 2D) = 0$

$2C - 4D = 1$

$+5D = -1$, $D = -\frac{1}{5}$

$2C = \frac{1}{5}$, $C = \frac{1}{10}$

$$y = \mathcal{L}^{-1}\left\{-\frac{17}{20}\cdot\frac{1}{s-3} + \frac{7}{4}\cdot\frac{1}{s-1} + \frac{1}{10}\frac{s}{s^2+1} - \frac{1}{5}\frac{1}{s^2+1}\right\}$$

$$= \boxed{-\frac{17}{20}e^{+3t} + \frac{7}{4}e^{t} + \frac{1}{10}\cos(t) - \frac{1}{5}\sin(t)}$$