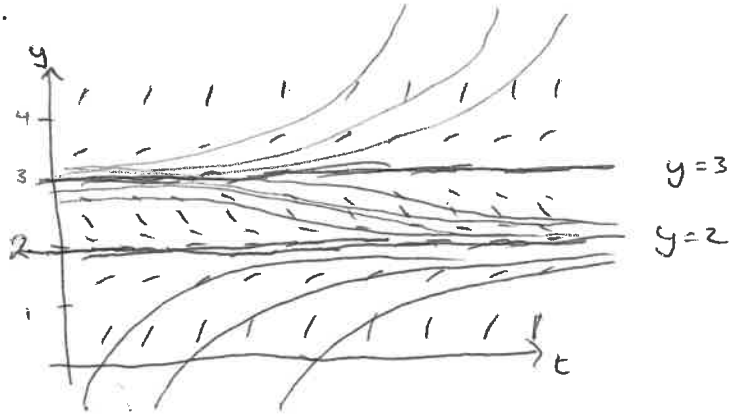
Name: Solutions

Directions: Show all work. No credit for answers without work.

1. Consider the differential equation $\frac{dy}{dt} = y^2 - 5y + 6$.

   (a) **[2 points]** Draw a direction field for $y(t)$.

   $\frac{dy}{dt} = y^2 - 5y + 6 = (y-2)(y-3)$



   (b) **[1 point]** Determine the limiting behavior of y as $t \to \infty$.

   If $y(0) < 3$, then $y \to 2$ as $t \to \infty$

   If $y(0) = 3$, then $y(t) = 3$ for all t.

   If $y(0) > 3$, then $y \to \infty$ as $t \to \infty$

2. **[3 points]** Find the general solution to $y' = 4 - 3y$.

$$y' = -3\left(y - \frac{4}{3}\right)$$

$$\frac{1}{y - \frac{4}{3}} \frac{dy}{dt} = -3$$

$$\int \frac{1}{y - \frac{4}{3}} \frac{dy}{dt}\, dt = \int -3\, dt$$

$$\int \frac{1}{y - 4/3}\, dy = -3t + C$$

$$\ln\left|y - 4/3\right| = -3t + C$$

$$y - 4/3 = Ce^{-3t}$$

$$\boxed{y = \frac{4}{3} + Ce^{-3t}}$$

3. Let $y(t)$ be the number of rabbits on an island at time t (months). The rabbits produce new offspring at a rate proportional to the population, with proportionality constant 2 (months)$^{-1}$. Owls hunt the rabbits, consuming a total of 100 rabbits per month.

(a) [**1 point**] Give a differential equation for $y(t)$.

$$\boxed{\frac{dy}{dt} = 2y - 100}$$

(b) [**2 points**] Given that the island starts with 45 rabbits, find a formula for $y(t)$.

$$\frac{dy}{dt} = 2(y-50)$$

$$\int \frac{1}{y-50}\,dy = \int 2\,dt$$

$$\ln|y-50| = 2t + C$$

$$y - 50 = Ce^{2t}$$

$$y = 50 + Ce^{2t}.$$

Impose $y(0) = 45$

$$45 = 50 + Ce^{2\cdot 0}$$

$$C = -5.$$

$$\boxed{y = 50 - 5e^{2t}}$$

(c) [**1 point**] Will the rabbits survive? If not, then how long will the rabbits last?

Since $y \to -\infty$ as $t \to \infty$, eventually the rabbits die out.

To find how long it takes, set $y(t) = 0$ and solve for t.

$$0 = 50 - 5e^{2t}$$

$$5e^{2t} = 50$$

$$e^{2t} = 10$$

$$2t = \ln(10)$$

$$t = \boxed{\tfrac{1}{2}\ln(10)} \approx \boxed{1.15 \text{ months}}$$