Short Answer [50 points]

Directions: Solve 2 of the following 4 problems. Include the work that supports your answers.

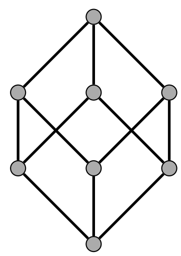
1. Let G_n be the graph whose vertex set is $\{(x,y) \in \mathbb{Z}^2 \colon 1 \le x, y \le n\}$ with (x,y) adjacent to (x',y') if and only if ($x < x'$ and $y < y'$) or ($x > x'$ and $y > y'$).

   (a) Draw G_1, G_2, and G_3.

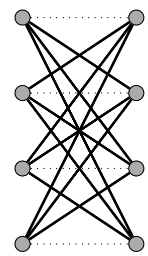
   (b) Determine the maximum size of a clique in G_n.

   Note: a complete answer to part (b) includes both a lower and an upper bound.

2. Show that the 3-dimensional cube Q_3 is isomorphic to the graph F obtained from $K_{4,4}$ by deleting 4 edges with distinct endpoints. (Hint: what are the 3-regular bipartite graphs on 8 vertices?)



Q_3



F

3. For each $n \ge 3$, construct an n-vertex graph with $3(n-2)$ edges in which every pair of cycles contains a common vertex.

4. Either find a decomposition of Q_3 into copies of the 4-vertex path P_4 or show that no such decomposition exists.

Deeper Questions [50 points]

Directions: Solve 2 of the following 4 problems.

1. A graph G is *splittable* if G has a spanning subgraph H such that $d_H(v) = d_G(v)/2$ for each vertex v. Prove that a connected graph G is splittable if and only if all vertices in G have even degree and G has an even number of edges.

2. Let $n \ge 2$. Prove that if G is an n-vertex graph in which every pair of cycles shares a common vertex, then $|E(G)| \le 3(n-2) + 1$.

   Comment: compare with the construction in Short Answer #3.

3. Let d be a graphic sequence, where $d = (d_1, \ldots, d_n)$ and $n \ge 2$. Prove that there is a **connected** graph that realizes d if and only if $\sum_{i=1}^{n} d_i \ge 2(n-1)$ and each d_i is positive.

4. Let T be a strongly connected n-vertex tournament. Prove that T contains a directed k-cycle for each k with $3 \le k \le n$.