Directions: Solve 3 of the following 4 problems. All written work must be your own, using only permitted sources. See the "General Guidelines and Advice" on the homework page for more details.

1. Let G be a k-colorable graph, and let P be a set of vertices in G such that $\text{dist}(x, y) \geq 4$ whenever $x, y \in P$. Prove that every coloring of P with colors from $[k+1]$ extends to a proper $(k+1)$-coloring of G.

2. Prove that if G has no induced $2K_2$, then $\chi(G) \leq \binom{\omega(G)+1}{2}$. (Hint: use a maximum clique to define a collection of $\binom{\omega(G)}{2} + \omega(G)$ independent sets that cover the vertices.)

3. Let G be a k-chromatic graph with girth 6 and order n. Construct G' as follows. Let T be an independent set of kn vertices. Take $\binom{kn}{n}$ pairwise disjoint copies of G, one for each way to choose an n-set $S \subset T$. Add a matching between each copy of G and its corresponding n-set S. Prove that the resulting graph has chromatic number $k+1$ and girth 6. (Comment: Since C_6 has chromatic number 2 and girth 6, the process can start and these graphs exist.)

4. Let G be a graph with no induced copy of the claw ($K_{1,3}$).

   (a) Show that in a proper coloring, each subgraph of G induced by the union of two color classes consists of paths and even cycles.

   (b) An *equitable* coloring of a graph is a proper vertex-coloring in which every pair of color classes differs in size by at most 1. Prove that G has an equitable coloring that is optimal (i.e. uses just $\chi(G)$ colors).