Directions: Solve the following problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. Let p be an odd prime. Determine the number of mutually incongruent solutions to $x^2+y^2 \equiv 0 \pmod{p}$. (A solution (x, y) is congruent to (x', y') if $(x, y) \equiv (x', y') \pmod{p}$. When $p = 3$, there is 1 solution $(0, 0)$, and when $p = 5$, there are 9 solutions.)

2. *Infinitely many primes congruent to* 7 *modulo* 8*.*
   (a) Prove that if n is an integer and p is an odd prime dividing $n^2 - 2$, then $p \equiv \pm 1 \pmod{8}$.
   (b) Prove that there are infinitely many primes p such that $p \equiv 7 \pmod{8}$.

3. Sums of three squares.
   (a) [NT 11-2.9] Show that no integer of the form $4^a(8m + 7)$ is the sum of three squares. Hint: consider the congruence $x^2 + y^2 + z^2 \equiv 7 \pmod{8}$.
   (b) Prove or disprove: if x and y are representable as the sum of three squares, then so is xy.
   (c) Prove that if x is representable as the sum of three *positive* squares, then so is x^2.

4. Partition Exercises.
   (a) Find the conjugate partition to $16 = 5 + 4 + 4 + 2 + 1$.
   (b) [NT 12-3.1] For the case $n = 8$, list the corresponding pairs of partitions of n in which all parts are odd and partitions of n into distinct parts given by Theorem 12-3.

5. Let $P(q)$ be the generating function for the partition numbers. That is, $P(q) = \sum_{n \geq 0} p(n)q^n$ by definition, and $P(q) = \prod_{j \geq 1} \frac{1}{1-x^j}$ for $|q| < 1$ by Theorem 13–3.
   (a) Let $a_k(n)$ be the number of partitions of n in which each part is used less than k times, and let $A_k(q)$ be the generating function $A_k(q) = \sum_{n \geq 0} a_k(n)q^n$. Show that $A_k(q) = \frac{P(q)}{P(q^k)}$ for $|q| < 1$.
   (b) Let $b_k(n)$ be the number of partitions of n in which no part is divisible by k, and let $B_k(q)$ be the generating function $B_k(q) = \sum_{n \geq 0} b_k(n)q^n$. Show that $B_k(q) = \frac{P(q)}{P(q^k)}$ for $|q| < 1$.

   Note: Since $A_k(q) = B_k(q)$, it follows that $a_k(n) = b_k(n)$ for all n.

6. **[Challenge]** Prove or disprove: there are infinitely many integer pairs (a, b) such that
$$2a^2 - b^2 = 1.$$

7. **[Challenge]** The Fibonacci sequence is defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Prove that for each positive integer m, there are infinitely many Fibonacci numbers that are divisible by m.