Directions: Solve the following problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. Prove that p is the smallest prime that divides $(p-1)!+1$.

2. Primes.

   (a) Let p be prime and let $a \in \mathbb{Z}$. Prove that if $a^{p-1}+p-1$ is prime, then $|a|=1$ or $p \mid a$.

   (b) Conclude that if p, q, and $q^{p-1}+p-1$ are all prime, then $p=q$.

3. [NT 5-2.4] Let $k=\phi(m)$. Prove that if $r_1,\ldots,r_k$ is a reduced residue system modulo m and m is odd, then $r_1+r_2+\cdots+r_k \equiv 0 \pmod{m}$.

4. [NT 5-3.4]

   (a) Prove that for each n, there are n consecutive integers, each of which is divisible by a perfect square larger than 1.

   (b) Using your proof above, explicitly find 3 consecutive integers, each of which is divisible by a perfect square larger than 1. In your answer, give the integers as well as the corresponding perfect squares.

5. [NT 5-4.1] Find the set of solutions to the following system of congruences:

$$\begin{aligned} 2x &\equiv 1 \pmod{5} \\ 3x &\equiv 9 \pmod{6} \\ 4x &\equiv 1 \pmod{7} \\ 5x &\equiv 9 \pmod{11} \end{aligned}$$

6. Let $A=\{a^2-b^2 \colon a,b \in \mathbb{Z}\}$. Give a simple characterization of A (with proof of correctness).

7. **[Challenge]** Prove that if $n \geq 2$, then the sum $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$ is not an integer.