Directions: Solve 5 of the following 6 problems. All written work must be your own, using only permitted sources. See the "General Guidelines and Advice" on the homework page for more details.

1. Let v be a vertex of a 2-connected graph G. Prove that v has a neighbor u such that $G-u-v$ is connected. Find a 2-edge-connected graph G that has a vertex v such that for each neighbor u of v, the graph $G-u-v$ is disconnected.

2. Let x and y be vertices in a 3-connected graph G. Show that there is an induced xy-path P such that $G-V(P)$ is connected.

3. Let G be a $2k$-edge-connected graph with at most two vertices of odd degree. Prove that G has a k-edge-connected orientation. (Recall that a digraph D is k-edge-connected if $|[S,\overline{S}]| \geq k$ when S is a nonempty proper subset of $V(D)$. Here, the directed cut $[S,\overline{S}]$ is the set of all edges from vertices in S to vertices outside S.)

4. Minimally k-edge-connected graphs.

   (a) For $S \subseteq V(G)$, let $d(S) = |[S,\overline{S}]|$. Let X and Y be nonempty proper vertex subsets of G. Prove that $d(X\cap Y) + d(X \cup Y) \leq d(X) + d(Y)$. Hint: the sets $X \cap Y$, $X - Y$, $Y - X$, and $\overline{X} \cap \overline{Y}$ partition $V(G)$. Draw a picture in which $V(G)$ is organized by this partition and consider contributions from various types of edges.

   (b) A k-edge-connected graph G is *minimally k-edge-connected* if, for each edge e in G, the graph $G-e$ is not k-edge-connected. Prove that $\delta(G) = k$ when G is minimally k-edge-connected. Hint: Consider a minimal set S such that $|[S,\overline{S}]| = k$. If $|S| \neq 1$, then use $G-e$ for some $e \in E(G[S])$ to obtain another set T with $|[T,\overline{T}]| = k$ such that S,T contradict part (a).

5. Use network flows to prove Menger's Theorem for edge-disjoint paths in graphs: $\kappa'(x,y) = \lambda'(x,y)$. (Recall that $\kappa'(x,y)$ is the minimum size of a set of edges S such that $G-S$ has no xy-path, and $\lambda'(x,y)$ is the maximum size of a set of edge-disjoint xy-paths.)

6. Let G be a graph whose odd cycles are pairwise intersecting, meaning that every two odd cycles in G have a common vertex. Prove that $\chi(G) \leq 5$.