Directions: Solve 5 of the following 6 problems. All written work must be your own, using only permitted sources. See the "General Guidelines and Advice" on the homework page for more details.

1. Let $d_1, \ldots, d_n$ be positive integers with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, \ldots, d_n$ if and only if $\sum d_i = 2n - 2$.

2. For $n \geq 4$, let G be an n-vertex graph with at least $2n-3$ edges. Prove that G has two cycles of equal length.

3. Tree Subgraphs

   (a) Let G be a connected graph on at least $k+1$ vertices such that $d(u) + d(v) \geq 2k-1$ whenever u and v have distance 2 in G. Prove that if T is a tree with k edges, then T is a subgraph of G. Hint: for each j with $0 \leq j \leq k$ and each tree T_j with j edges, obtain a copy of T_j in G in which each vertex in T_j has a prescribed number of neighbors outside T_j.

   (b) For $k \geq 3$, give an example of a connected graph G on at least $k+1$ vertices such that $d(u) + d(v) \geq 2k-2$ and some tree with k edges fails to be a subgraph of G.

   Comment: Since $\delta(G) \geq k$ implies that each component of G satisfies the hypotheses in (a), this strengthens Proposition 2.1.8.

4. Use Cayley's Formula to prove that the graph obtained from K_n by deleting an edge has $(n-2)n^{n-3}$ spanning trees.

5. Let G be an (X, Y)-bigraph. A *near-matching* in G is a set of edges M such that each vertex in X is the endpoint of at most one edge in M and each vertex in Y is the endpoint of at most two edges in M. Find and prove an analogue of Hall's theorem that characterizes when G has a near-matching saturating X. (You may use Hall's theorem in the proof of your new characterization.)

6. Two people play a game on a graph G, alternately choosing distinct vertices. Player 1 starts by choosing any vertex. Each subsequent choice must be adjacent to the preceding choice (of the other player). Thus together they follow a path. The last player able to move wins.

   Prove that the second player has a winning strategy if G has a perfect matching, and otherwise the first player has a winning strategy. (Hint: for the second part, the first player should start with a vertex omitted by some maximum matching.)