Directions: Solve 5 of the following 6 problems. All written work must be your own, using only permitted sources. See the "General Guidelines and Advice" on the homework page for more details.

1. [IGT 1.3.22] Let G be a nonbipartite triangle-free graph with n vertices and minimum degree k. Let l be the minimum length of an odd cycle in G.
   (a) Let C be a cycle of length l in G. Prove that every vertex not in $V(C)$ contains at most two neighbors in $V(C)$.
   (b) By counting the edges joining $V(C)$ and $V(G) - V(C)$ in two ways, prove that $n \geq kl/2$ (and thus $l \leq 2n/k$).
   (c) When k is even, prove that the inequality of part (b) is best possible. (Hint: form a graph having $k/2$ pairwise disjoint l-cycles.)
2. [IGT 1.3.44] Let G be a graph with average degree a. Recall that $a = 2|E(G)|/|V(G)|$.
   (a) Prove that $G - x$ has average degree at least a if and only if $d(x) \leq a/2$.
   (b) Prove that if $a > 0$, then G has a subgraph with minimum degree greater than $a/2$. Hint: what happens if we delete a vertex of degree at most $a/2$?
   (c) Show that there is no constant c greater than $1/2$ such that G must have a subgraph with minimum degree greater than ca; this proves that the bound in part (b) is best possible. Hint: Use $K_{1,n-1}$.
3. [IGT 1.4.25]
   (a) Prove that every connected graph has an orientation in which the number of vertices with odd outdegree is at most 1. (Hint: consider an orientation with the fewest number of vertices with odd outdegree.)
   (b) Use part (a) to conclude that every connected graph with an even number of edges has a P_3-decomposition.
4. [IGT 1.4.29] Suppose that G is a graph and D is an orientation of G that is strongly connected. Prove that if G has an odd cycle, then D has a (directed) odd cycle. (Hint: consider each pair $\{v_i, v_{i+1}\}$ in an odd cycle $v_1 \cdots v_k$ of G.)
5. [IGT 1.4.34] Let G and H be tournaments on the same vertex set. Prove that $d^+_G(v) = d^+_H(v)$ for each vertex v if and only if G can be turned into H by a sequence of direction-reversals on cycles of length 3.
6. Let G be a directed graph without loops. Prove that G has an independent set S such that every vertex in G is reachable from a vertex in S by a directed path of length at most 2. Hint: use induction on $|V(G)|$ and recall that the induction hypothesis applies to all graphs with fewer vertices, not just graphs with $|V(G)| - 1$ vertices.