**Directions:** Solve the following problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. [NT 9-3.3] Prove that if $a$ and $c$ are positive, odd, and relatively prime, then $\left(\frac{a}{c}\right) = \left(\frac{c}{a}\right)$ unless $a \equiv c \equiv 3 \pmod 4$, in which case $\left(\frac{a}{c}\right) = -\left(\frac{c}{a}\right)$. Note: here, $\left(\frac{a}{c}\right)$ and $\left(\frac{c}{a}\right)$ are Jacobi symbols.

2. [NT 9-3.6] Let $m$ be an odd, positive integer. Is it possible that the Jacobi symbol $\left(\frac{n}{m}\right)$ satisfies $\left(\frac{n}{m}\right) = 1$ but $x^2 \equiv n \pmod m$ has no solution? Prove your answer.

3. [NT 9-4] Determine (with proof) whether the following congruences have solutions.

   (a) $x^2 \equiv 17 \pmod{29}$

   (b) $3x^2 \equiv 12 \pmod{23}$

   (c) $2x^2 \equiv 27 \pmod{41}$

   (d) $x^2 + 5x \equiv 12 \pmod{31}$ *Hint:* complete the square.

   (e) $x^2 \equiv 19 \pmod{30}$

4. Let $p$ be a prime.

   (a) Let $a$ be an integer such that $p \nmid a$, and let $h$ be the order of $a$. Show that if $a \not\equiv 1 \pmod p$, then $1 + a + a^2 + \cdots + a^{h-1} \equiv 0 \pmod p$.

   (b) Let $Q = \{a\colon\ 1 \le a \le p-1 \text{ and } a \text{ is a quadratic residue}\}$. Prove that if $p \ge 5$, then $\sum_{t \in Q} t \equiv 0 \pmod p$.

5. Let $m$ and $n$ be positive integers. Prove that $\gcd(2^m - 1, 2^n - 1) = 2^d - 1$, where $d = \gcd(m, n)$. *Comment*: except for $n \in \{1, 6\}$, there is a prime that divides $2^n - 1$ but divides no integer in $\{2^m - 1\colon\ 1 \le m < n\}$. Can you find a way to prove this? It may be hard.

6. **[Challenge]** Let $p$ be a prime and let $R = \{a\colon\ 1 \le a \le p-1 \text{ and } a \text{ is a primitive root}\}$. Prove that $\sum_{t \in R} t \equiv \mu(p-1) \pmod p$, where $\mu(n)$ is the Möbius function.