Directions: Solve the following problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. [NT 8-1.12] Let $\theta(x)$ be the sum of the natural logarithms of all the primes not exceeding x. Prove that $\theta(x) \leq \pi(x) \ln x$.

2. [NT 8-1.16] Let $n = 132!$. How many zeros are at the end of the base 2 representation of n? How many zeros are at the end of the base 10 representation of n?

3. [NT 9-1.1(c)] Use Euler's Criterion to determine whether 3 is a quadratic residue modulo 11. Note: use Euler's Criterion, not quadratic reciprocity or other techniques.

4. [NT 9-3.5] Use the Quadratic Reciprocity Law to prove that

$$\left(\frac{3}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \text{ or } 11 \pmod{12} \\ -1 & \text{if } p \equiv 5 \text{ or } 7 \pmod{12} \end{cases}$$

   for each prime p where $p \geq 5$.

5. [NT 9-4] Determine (with proof) whether the following congruences have solutions.
   (a) $x^2 \equiv 17 \pmod{29}$
   (b) $3x^2 \equiv 12 \pmod{23}$
   (c) $2x^2 \equiv 27 \pmod{41}$
   (d) $x^2 + 5x \equiv 12 \pmod{31}$ *Hint:* complete the square.
   (e) $x^2 \equiv 19 \pmod{30}$

6. A prime p is *lonely* in a if $p \mid a$ but $p^2 \nmid a$. A number a is *special* if it has no lonely primes.
   (a) Find a pair of consecutive special numbers.
   (b) Prove that there are infinitely many pairs of consecutive special numbers. *Hint*: given a pair $\{a, a+1\}$ of special numbers, construct another pair $\{b, b+1\}$ of special numbers with $b > a$.
   (c) Execute your proof to obtain two more pairs of consecutive special numbers. Note: the purpose of this part is for you to check your proof in part (b), so we do not want just any old pairs. We want the ones your proof produces.
   (d) Prove that there are no intervals of 4 special numbers. That is, prove that at least one integer in $\{a, a+1, a+2, a+3\}$ is not special.

   Remark: this naturally begs the question: are there consecutive blocks of special numbers of size 3? I do not know the answer to this question, but numerical evidence from sage suggests the answer may be no.

7. **[Challenge]** Let A be the set of all integers n such that n divides $3^n - 1$.
   (a) Prove that if m and n are in A, then $\gcd(m, n) \in A$.
   (b) Prove that if $n \in A$ and p is a prime that divides n, then $np \in A$.
   (c) Prove that if $n \in A$ and p is the largest prime dividing n, then $n/p \in A$. Hint: express n as $n = mp^k$ where $p \nmid m$ and study the order of 3^m modulo mp^{k-1}.