Math545 **Homework 3** **Due Tues, Sept. 29, 2015**

Directions: Solve the following problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. [NT 3-3.1] Prove that p is the smallest prime that divides $(p-1)!+1$.

2. Let $f(n)$ be the number of representations of n as the sum of square numbers. For example, since $25 = 0^2 + 5^2$ and $25 = 3^2 + 4^2$, it follows that $f(25) \geq 2$. In fact, $f(25) = 2$ and 25 is the least integer n such that $f(n) \geq 2$. Similarly, $f(12) = 0$ since there is no way to write 12 as the sum of two squares. On the other hand, $f(18) = 1$ because the only way to express 18 as the sum of two squares is $18 = 3^2 + 3^2$.

   (a) Determine the least integer n such that $f(n) \geq 3$. What are the representations of n?

   (b) Determine the least integer n such that $f(n) \geq 20$. What are the representations of n? *[Hint: Using an algorithm computes $f(n)$ in a loop starting with $n = 1$ and working upward will probably be too slow. To be more efficient, try computing all the values $f(1), f(2), \ldots, f(n)$ all at once for a trial value of n. If we find any numbers with 20 distinct representations, we're done: just grab the smallest. Otherwise, double n and try again.]*

3. [NT 5-2.4] Let $k = \phi(m)$. Prove that if $r_1, \ldots, r_k$ is a reduced residue system modulo m and m is odd, then $r_1 + r_2 + \cdots + r_k \equiv 0 \pmod{m}$.

4. [NT 5-3.4]

   (a) Prove that for each n, there are n consecutive integers, each of which is divisible by a perfect square larger than 1.

   (b) Using your proof above, explicitly find 3 consecutive integers, each of which is divisible by a perfect square larger than 1. In your answer, give the integers as well as the corresponding perfect squares.

5. [NT 5-4.1] Find the set of solutions to the following system of congruences:

$$\begin{aligned} 2x &\equiv 1 \pmod{5} \\ 3x &\equiv 9 \pmod{6} \\ 4x &\equiv 1 \pmod{7} \\ 5x &\equiv 9 \pmod{11} \end{aligned}$$

6. Let $A = \{a^2 - b^2 \colon\ a, b \in \mathbb{Z}\}$. Give a simple characterization of A (with proof of correctness).

7. **[Challenge]** Let $m = 2^r 3^s$ where r and s are positive integers. Prove that $m \mid a^m - 1$ if and only if $\gcd(a, m) = 1$.

8. **[Challenge]** Prove that if $n \geq 2$, then the sum $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is not an integer.