Directions: Solve the following problems; computer problems (marked with a icon) and challenge problems are optional. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. [NT 2-2.4] The *least common multiple* of a and b, denoted $\operatorname{lcm}(a, b)$, is the smallest positive integer ℓ such that $a \mid \ell$ and $b \mid \ell$. Prove that if a and b are positive integers, then $\operatorname{lcm}(a, b) = ab/\gcd(a, b)$.

2. [NT 2-3.1] Find the general solution (if solutions exist) of each of the following linear Diophantine equations:
   (a) $15x + 51y = 41$
   (b) $23x + 29y = 25$
   (c) $121x - 88y = 572$

3. Binomial Coefficients and Parity I.
   (a) [NT 3-1.3] Using the definition of $\binom{n}{r}$, show combinatorially that $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.
   (b) Prove that if n is even and r is odd, then $\binom{n}{r}$ is even.

4. Binomial Coefficients and Parity II. Pascal's Triangle is an arrangement of the binomial coefficients in which $\binom{n}{k}$ is placed at position k in row n. For example, the rows $n = 0$ through $n = 6$ of Pascal's Triangle are shown on the left:

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

1	1	1	1	1	1	1
1	2	3	4	5	6	
1	3	6	10	15		
1	4	10	20			
1	5	15				
1	6					
1						

   For the computer problem, it is convenient to rotate Pascal's Triangle so that $\binom{x+y}{x}$ appears in position (x, y), as shown to the right. Write a program whose output is a (30×30)-array of characters with an asterisk in position (x, y) if $\binom{x+y}{x}$ is even and a blank space otherwise. How does the grid look?

5. Binomial Coefficients and Parity III. A positive integer n is *excellent* if the set $\left\{\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}\right\}$ contains only odd integers.
   (a) Which numbers are excellent? Based on examining data, formulate a hypothesis.
   (b) **[Harder Challenge]** Prove that your hypothesis is correct. Hint: determine the maximum integer t such that 2^t divides $n!$.

6. [NT 3-2.3] Prove that n^5 and n have the same last digit.

7. Prove that if a and b are positive integers, then it is not possible for both $a + b^2$ and $a^2 + b$ to be square numbers (i.e. of the form k^2 for some integer k). Hint: after a^2, what is the next largest square?

8. **[Easier Challenge]** Prove that if n is an integer and $n \geq 2$, then $n^4 + 4^n$ is not prime.