Directions: Solve 5 of the following 6 problems. All written work must be your own, using only permitted sources. See the "General Guidelines and Advice" on the homework page for more details.

1. [IGT 4.1.23] Connectivity and perfect matchings.

   (a) Let G be an r-connected graph of even order having no $K_{1,r+1}$ as an induced subgraph. Prove that G has a 1-factor.

   (b) For each r, construct an r-connected graph of even order that does not contain an induced copy of $K_{1,r+3}$ and has no 1-factor.

   (Comment: this leaves unresolved whether every r-connected graph of even order without an induced copy of $K_{1,r+2}$ has a 1-factor.)

2. [IGT 4.1.25] Let G be a simple graph with diameter 2, and let $[S, \overline{S}]$ be a minimum edge cut with $|S| \leq |\overline{S}|$.

   (a) Prove that every vertex of S has a neighbor in $\overline{S}$.

   (b) Use part (a) and Corollary 4.1.13 (i.e. $|S| > \delta(G)$ when $|[S, \overline{S}]| < \delta(G)$ and S is a nonempty proper subset of $V(G)$) to prove that $\kappa'(G) = \delta(G)$ when G has diameter 2.

3. [IGT 4.2.15] Let v be a vertex of a 2-connected graph G. Prove that v has a neighbor u such that $G - u - v$ is connected. Find a 2-edge-connected graph G that has a vertex v such that for each neighbor u of v, the graph $G - u - v$ is disconnected.

4. [IGT 4.2.21] Let G be a $2k$-edge-connected graph with at most two vertices of odd degree. Prove that G has a k-edge-connected orientation. (Recall that a digraph D is k-edge-connected if $|[S, \overline{S}]| \geq k$ when S is a nonempty proper subset of $V(D)$.)

5. [IGT 4.2.{36,37}] Minimally k-edge-connected graphs.

   (a) For $S \subseteq V(G)$, let $d(S) = |[S, \overline{S}]|$. Let X and Y be nonempty proper vertex subsets of G. Prove that $d(X \cap Y) + d(X \cup Y) \leq d(X) + d(Y)$. Hint: the sets $X \cap Y$, $X - Y$, $Y - X$, and $\overline{X} \cap \overline{Y}$ partition $V(G)$. Draw a picture in which $V(G)$ is organized by this partition and consider contributions from various types of edges.

   (b) A k-edge-connected graph G is *minimally k-edge-connected* if, for each edge e in G, the graph $G - e$ is not k-edge-connected. Prove that $\delta(G) = k$ when G is minimally k-edge-connected. Hint: Consider a minimal set S such that $|[S, \overline{S}]| = k$. If $|S| \neq 1$, then use $G - e$ for some $e \in E(G[S])$ to obtain another set T with $|[T, \overline{T}]| = k$ such that S, T contradict part (a).

6. [IGT 4.3.6] Use network flows to prove Menger's Theorem for edge-disjoint paths in graphs: $\kappa'(x, y) = \lambda'(x, y)$. (Recall that $\kappa'(x, y)$ is the minimum size of a set of edges S such that $G - S$ has no xy-path, and $\lambda'(x, y)$ is the maximum size of a set of edge-disjoint xy-paths.)