Directions: Solve 5 of the following 6 problems. All written work must be your own, using only permitted sources. See the "General Guidelines and Advice" on the homework page for more details.

1. [IGT 3.1.21] Let G be an (X,Y)-bigraph such that $|N(S)| > |S|$ when $\varnothing \neq S \subsetneq X$. Prove that every edge of G is contained in a matching that saturates X.

2. [IGT 3.1.25] A *doubly stochastic matrix* Q is a nonnegative real matrix in which every row and every column sums to 1. Prove that a doubly stochastic matrix Q can be expressed as $Q = c_1 P_1 + \cdots + c_m P_m$ where $c_1, \ldots, c_m$ are nonnegative real numbers summing to 1 and $P_1, \ldots, P_m$ are permutation matrices. For example,

$$\begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/6 & 5/6 \\ 1/2 & 1/2 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

   Hint: Use induction on the number of nonzero entries in Q.

3. [IGT 3.1.27] A *positional game* consists of a set X of positions $x_1, \ldots, x_n$ and a family $W_1, \ldots, W_m$ of winning sets of positions (Tic-Tac-Toe has nine positions and eight winning sets). Two players alternately choose positions; a player wins by collecting a winning set.

   Suppose that each winning set has size at least a and each position appears in at most b winning sets (in Tic-Tac-Toe, $a = 3$ and $b = 4$). Prove that Player 2 can force a draw if $a \geq 2b$. (Hint: Form an (X,Y)-bigraph G, where $Y = \{w_1, \ldots, w_m\} \cup \{w'_1, \ldots, w'_m\}$, with edges $x_i w_j$ and $x_i w'_j$ whenever $x_i \in W_j$. How can Player 2 use a matching in G?) Comment: This result implies that Player 2 can force a draw in d-dimensional Tic-Tac-Toe.

4. [IGT 3.1.30] Determine the maximum number of edges in a bipartite graph that contains no matching with k edges and no star with l edges.

5. [IGT 3.3.16] Let G be a k-regular graph of even order that remains connected whenever $k-2$ edges are deleted. Prove that G has a 1-factor.

6. [IGT 3.3.20] Prove that a 3-regular graph has a 1-factor if and only if it decomposes into copies of P_4.