Directions: Solve 5 of the following 6 problems. All written work must be your own, using only permitted sources. See the "General Guidelines and Advice" on the homework page for more details.

1. [IGT 2.1.27] Let $d_1, \ldots, d_n$ be positive integers with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, \ldots, d_n$ if and only if $\sum d_i = 2n - 2$.

2. [IGT 2.1.41] For $n \geq 4$, let G be an n-vertex graph with at least $2n - 3$ edges. Prove that G has two cycles of equal length.

3. [IGT 2.1.63] Prove that every n-vertex graph with $n+1$ edges has girth at most $\lfloor (2n+2)/3 \rfloor$. For each n, construct an example achieving this bound.

4. [IGT 2.1.72] Prove that if $G_1, \ldots, G_k$ are pairwise-intersecting subtrees of a tree G, then G has a vertex that belongs to all of $G_1, \ldots, G_k$. Hint: use induction on k.

5. [IGT 2.2.7] Use Cayley's Formula to prove that the graph obtained from K_n by deleting an edge has $(n-2)n^{n-3}$ spanning trees.

6. [IGT 3.1.18] Two people play a game on a graph G, alternately choosing distinct vertices. Player 1 starts by choosing any vertex. Each subsequent choice must be adjacent to the preceding choice (of the other player). Thus together they follow a path. The last player able to move wins.

   Prove that the second player has a winning strategy if G has a perfect matching, and otherwise the first player has a winning strategy. (Hint: for the second part, the first player should start with a vertex omitted by some maximum matching.)