

Directions: Solve 5 of the following 6 problems. All written work must be your own, using only permitted sources. See the “General Guidelines and Advice” on the homework page for more details.

1. [IGT 1.2.{17,27}]
 - (a) Let G_n be the graph whose vertices are the permutations of $\{1, \dots, n\}$ with two permutations adjacent if they differ by interchanging a pair of adjacent entries. Note that $G_3 = C_6$. Prove that G_n is connected.
 - (b) Let H_n be the graph whose vertices are the permutations of $\{1, \dots, n\}$ with two permutations adjacent if they differ by interchanging a pair of entries (which may or may not be adjacent). Note that $G_3 = K_{3,3}$ and G_n is a subgraph of H_n . Prove that H_n is bipartite. Hint: for each permutation $a_1 \cdots a_n$, count the pairs (i, j) with $i < j$ and $a_i > a_j$; these are called *inversions*.
2. [IGT 1.2.{29,42}]
 - (a) Let G be a connected graph not having P_4 or C_3 as an induced subgraph. Prove that G is a biclique.
 - (b) Let G be a connected graph not having P_4 or C_4 as an induced subgraph. Prove that G has a vertex adjacent to all other vertices. (Hint: consider a vertex of maximum degree.)
3. [IGT 1.2.38] Prove that every n -vertex graph with at least n edges contains a cycle.
4. [IGT 1.2.40] Let P and Q be paths of maximum length in a connected graph G . Prove that P and Q have a common vertex.
5. [IGT 1.3.{12,18}]
 - (a) Prove that an even graph has no cut-edge. For each $k \geq 1$, construct a $(2k + 1)$ -regular graph having a cut-edge.
 - (b) For $k \geq 2$, prove that a k -regular bipartite graph has no cut-edge.
6. [IGT 1.3.20] Count the cycles of length n in K_n and the cycles of length $2n$ in $K_{n,n}$.