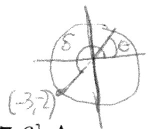
Solutions

1. Rewrite $-2\cos(3t) - 3\sin(3t)$ in the form $R\cos(\omega_0 t - \delta)$.

$R = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13}$; $\omega_0 = 3$.

$\tan(\delta) = \frac{-3}{-2} = \frac{3}{2}$



$\theta = \tan^{-1}\left(\frac{3}{2}\right)$

$\delta = \theta + \pi = \tan^{-1}\left(\frac{3}{2}\right) + \pi$

Exact!

$\boxed{\sqrt{13}\cos\left(3t - \left(\tan^{-1}\left(\tfrac{3}{2}\right) + \pi\right)\right)}$

Approx:

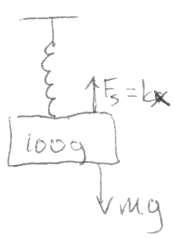
$\boxed{3.61\cos(3t - 4.12)}$

2. [3.7.6] A mass of 100 g stretches a spring 5 cm. If the mass is set in motion from its equilibrium position with a downward velocity of 10cm/s, and if there is no damping, (a) determine the position u of the mass at any time t. (b) When does the mass first return to its equilibrium position? (c) Determine the frequency, period, amplitude, and phase of the motion.

(a)

$\Rightarrow$ Units: g, cm, s

$mu'' + \gamma u' + ku = 0$

- $m = 100\,g$
- $\gamma = 0$ since motion is undamped
- 

$mg = kx$

$(100\,g)\,\frac{9.8\,m}{s^2} = (k)(5\,cm)$

$k = \frac{100}{5} \cdot 9.8 \cdot \frac{100\,cm}{1\,m} \cdot \frac{m}{s^2} \cdot \frac{g}{cm}$

$= 19{,}600\ \frac{g\,m}{s^2}$.

$100u'' + 0u' + 19{,}600 = 0$

$u'' + 196u = 0$

$r^2 + 196 = 0$

$r = \pm 14i$

$u(t) = c_1\cos(14t) + c_2\sin(14t)$

$u'(t) = -14c_1\sin(14t) + 14c_2\cos(14t)$

$u(0) = 0$: $\quad 0 = c_1 \cdot 1 + \cancel{c_2 \cdot 0}$ $\quad c_1 = 0$

$u'(0) = 10$: $\quad 10 = -14 \cdot \cancel{0} + 14c_2 \cdot 1$ $\quad c_2 = \frac{5}{7}$

$\boxed{u(t) = \frac{5}{7}\sin(14t), \ u \text{ in cm}; \ t \text{ in s.}}$

(b) Solve for t in $u(t) = 0$:

$0 = \frac{5}{7}\sin(14t)$

$\sin(14t) = 0$

$14t$ = integer multiple of π

First time: $14t = \pi$, $\boxed{t = \frac{\pi}{14} \approx 0.224\,s}$

(c) $R = \sqrt{0^2 + \left(\frac{5}{7}\right)^2} = \frac{5}{7}$

$(0, 5/7)$

$\delta = \pi/2$

(Note: $\tan(\delta) = \frac{5/7}{0}$ is undefined!)

$\boxed{\text{Phase} = \delta = \frac{\pi}{2}}$

So $u(t) = \frac{5}{7}\cos(14t - \pi/2)$.

$\boxed{\text{Amplitude} = R = \frac{5}{7}\,cm}$

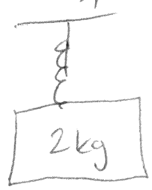
$\boxed{\text{Frequency} = \omega_0 = 14\ rad/s}$

$\boxed{\text{Period} = \frac{2\pi}{\omega_0} = \frac{2\pi}{14} \approx 0.449\,s}$

3.7.11 A spring is stretched 10 cm by a force of 3 N. (Note: one Newton, denoted N, is $1\text{kg} \cdot \text{m/s}^2$.) A mass of 2 kg is hung from the spring and is also attached to a viscous damper that exerts a force of 3 N when the velocity of the mass is 5 m/s. If the mass is pulled down 5 cm below its equilibrium position and given an initial downward velocity of 10 cm/s, (a) determine its position u at any time t. (b) Find the quasi frequency μ and the ratio of μ to the natural frequency of the corresponding undamped motion.

Units: kg, m, s.

(a)



$$mu'' + \gamma u' + ku = 0$$

① $m = 2\text{kg}$

② $F_s = kx$

$$3\,\frac{\text{kg m}}{s^2} = k \cdot \frac{1}{10}\,\text{m}$$

$$k = 30\,\frac{\text{kg}}{s^2}$$

③ $F_d = \gamma u'$

$$3\,\frac{\text{kg m}}{s^2} = \gamma \cdot 5\,\text{m/s}$$

$$\gamma = \frac{3}{5}\,\frac{\text{kg}}{s}$$

④ $2u'' + \frac{3}{5}u' + 30u = 0$

$$10u'' + 3u' + 150u = 0$$

$$10r^2 + 3r + 150 = 0$$

$$r = \frac{-3 \pm \sqrt{9 - 4\cdot 10\cdot 150}}{2\cdot 10} \approx -0.15 \pm 3.87i$$

$$\cancel{-\frac{3}{20} \pm \sqrt{\frac{9}{4} - 1500}} \quad \cancel{= -\frac{3}{2} \pm \sqrt{\frac{1591}{4}}\,i} \quad \cancel{\approx -1.5 \pm 19.94i}$$

So $u(t) = e^{-0.15t}\left(c_1\cos(3.87t) + c_2\sin(3.87t)\right)$

$$u'(t) = -0.15e^{-0.15t}\left(c_1\cos(3.87t) + c_2\sin(3.87t)\right)$$
$$+ e^{-0.15t}\left(-3.87c_1\sin(3.87t) + 3.87c_2\cos(3.87t)\right)$$

$u(0) = \frac{5}{100} = \frac{1}{20}$; $\frac{1}{20} = 1\left(c_1 + c_2\sin(0)\right)$; $c_1 = \frac{1}{20}$

$u'(0) = \frac{10}{100} = \frac{1}{10}$.

$$\frac{1}{10} = -0.15(1)(c_1\cdot 1) + (1)(0 + 3.87c_2)$$

$$\frac{1}{10} = -0.15\cdot\frac{1}{20} + 3.87c_2 \,;\quad c_2 = 0.028$$

(a) So $\boxed{u(t) \approx e^{-0.15t}\left(\frac{1}{20}\cos(3.87t) + 0.028\sin(3.87t)\right)}$ u in m, t in s

OR

$$R = \sqrt{\left(\tfrac{1}{20}\right)^2 + (0.028)^2} = 0.057 \qquad \boxed{u(t) \approx e^{-0.15t}\left(0.057\cos(3.87t - 0.51)\right)}$$

$$\delta = \tan^{-1}\left(\frac{0.028}{1/20}\right) \approx 0.51\text{ rad.}$$

Undamped: $2u'' + 30u = 0 \quad r = \pm\sqrt{15}\,i$

(b) $\boxed{\mu = 3.87\ \frac{\text{rad}}{\text{sec}}}$ Ratio: $\frac{\mu}{\omega_0} \approx \frac{3.87}{\sqrt{15}} \approx \boxed{0.9992}$