Solutions

Math261 **Worksheet 6** **September 30, 2014**

1. Solve $3y'' - 11y' - 4y = 0$ with $y(0) = 2$ and $y'(0) = 0$.

$3r^2 - 11r - 4 = 0$

$(3r+1)(r-4) = 0$

$3r+1=0$ or $r-4=0$

$r_1 = -\frac{1}{3}$ $\quad r_2 = 4$

$y_1 = e^{-\frac{1}{3}t}$ $\quad y_2 = e^{4t}$

So

$y = c_1 e^{-\frac{1}{3}t} + c_2 e^{4t}$

$y' = -\frac{1}{3}c_1 e^{-\frac{1}{3}t} + 4c_2 e^{4t}$

$\underline{y(0)=2}$: $2 = c_1 + c_2$

$\underline{y'(0)=0}$: $0 = -\frac{1}{3}c_1 + 4c_2$.

Solve for c_1, c_2:

$c_1 + c_2 = 2$

$-c_1 + 12c_2 = 0$

$13c_2 = 2$

$c_2 = \frac{2}{13}$

$c_1 = 2 - c_2 = \frac{26}{13} - \frac{2}{13} = \frac{24}{13}$

So

$$\boxed{y = \frac{24}{13}e^{-\frac{1}{3}t} + \frac{2}{13}e^{4t}}$$

2. Solve $y'' - 4y' + 5y = 0$ with $y(0) = 1$ and $y'(0) = -1$.

① $r^2 - 4r + 5 = 0$

$r = \frac{4 \pm \sqrt{16 - 4\cdot 1\cdot 5}}{2}$

$r = \frac{4 \pm \sqrt{-4}}{2}$

$r = \frac{4 \pm 2\sqrt{-1}}{2}$

$r = 2 \pm i$.

$r_1 = 2+i$ $\quad r_2 = 2-i$

$y_1 = e^{(2+i)t}$ $\quad \cancel{y_2 = e^{(2-i)t}}$ ← Discard

$= e^{2t} \cdot e^{ti}$

$= e^{2t}(\cos(t) + i\sin(t))$

$= e^{2t}\cos(t) + e^{2t}\sin(t)\, i$

② Use real & imaginary parts of y_1:

$y_3 = e^{2t}\cos(t)$ $\quad y_4 = e^{2t}\sin(t)$

③ $y = c_1 y_3 + c_2 y_4$

$y = e^{2t}(c_1\cos(t) + c_2\sin(t))$

$y' = 2e^{2t}(c_1\cos(t) + c_2\sin(t)) + e^{2t}(-c_1\sin(t) + c_2\cos(t))$

$= e^{2t}((2c_1 + c_2)\cos(t) + (2c_2 - c_1)\sin(t))$

$\underline{y(0)=1}$: $1 = 1\cdot(c_1 \cdot \cos(0) + c_2 \sin(0))$

$1 = c_1$

$\underline{y'(0)=-1}$: $-1 = 1\cdot((2c_1+c_2)\cdot\cos(0) + (2c_2 - c_1)\sin(0))$

$-1 = 2c_1 + c_2$; $\quad -1 = 2(1) + c_2$; $\quad c_2 = -3$.

So $\boxed{y = e^{2t}(\cos(t) - 3\sin(t))}$

3. Compute the Wronskian of the following groups of functions:

   (a) e^t, te^t

   (b) ~~$\sin^2 t$, $\cos^2 t$, 1~~

(a)

$$W = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix}$$

$$= e^t(e^t(1+t)) - (e^t)(te^t)$$

$$= e^{2t}(1+t) - te^{2t} = \boxed{e^{2t}}$$

(b) ~~$W = \begin{vmatrix} \sin^2 t & \cos^2 t & 1 \\ 2\sin t\cos t & & \end{vmatrix}$~~

4. [3.2.24] Verify that $y_1 = \cos 2t$ and $y_2 = \sin 2t$ are both solutions to $y'' + 4y = 0$. Do y_1 and y_2 form a fundamental set of solutions?

$y_1' = -2\sin(2t)$

$y_1'' = -4\cos(2t)$

$y_1'' + 4y_1 = -4\cos(2t) + 4\cos(2t)$

$= 0$ ✓

So y_1 is a soln.

$y_2' = 2\cos(2t)$

$y_2'' = -4\sin(2t)$

$y_2'' + 4y_2 = -4\sin(2t) + 4\sin(2t) = 0$ ✓

So y_2 is a soln

$$W = \begin{vmatrix} \cos 2t & \sin 2 \\ -2\sin 2t & 2\cos \end{vmatrix}$$

$= 2\cos^2 2t - (-2\sin^2 2t$

$= 2(\cos^2 2t + \sin^2 2$

$= 2 \cdot 1 = \boxed{2}$

So $\boxed{\text{yes}}$ y_1 and y_2 form a fundamental set

5. [Misc. Prac.] Solve $\frac{dy}{dx} = 3 - 6x + y - 2xy$.

Linear first order diff eqn.

$$\frac{dy}{dx} + (2x-1)y = 3 - 6x$$

$$\mu = e^{\int (2x-1)dx} = e^{x^2-x}$$

$$e^{x^2-x}\frac{dy}{dx} + (2x-1)e^{x^2-x}y = (3-6x)e^{x^2-x}$$

$$\frac{d}{dx}\left[e^{x^2-x}y\right] = (3-6x)e^{x^2-x}$$

$$e^{x^2-x}y = \int (3-6x)e^{x^2-x}dx$$

$$e^{x^2-x}y = \int \frac{3-6x}{2x-1}e^u du \qquad u = x^2 - x, \quad du = 2x - 1\,dx$$

$$e^{x^2-x}y = \int -3e^u du$$

$$e^{x^2-x}y = -3e^u + C$$

$$\boxed{y = -3 + Ce^{-(x^2-x)}}$$