Solutions.

(a) population after 2 years,

1. A population of ants grows logistically. Initially, the ant population is 10% of the carrying capacity. After 1 year, the ant population has doubled. Find (b) the time at which the population reaches 90% of carrying capacity and (c) the time at which the population is increasing fastest. *Hint:* recall the logistic equation $\frac{dy}{dt} = r(1-(y/K))y$.

Let $y(t)$ be the population of ants in units of carrying capacity so that $K=1$, $y(0)=.1$, and $y(1)=2y(0)=0.2$.

$$\frac{dy}{dt} = r(1-y)y$$

$$\frac{1}{(1-y)y}\frac{dy}{dt} = r$$

$$\int \frac{1}{(1-y)y}\,dy = \int r\,dt$$

$$\frac{A}{1-y}+\frac{B}{y} = \frac{1}{(1-y)y}$$

$$Ay + B(1-y) = 1$$

$y=0$: $B=1$

$y=1$: $A=1$

$$\int \frac{1}{1-y}\,dy + \int \frac{1}{y}\,dy = rt + C$$

$$-\ln|1-y| + \ln|y| = rt + C$$

$$\ln\left|\frac{y}{1-y}\right| = rt + C$$

$$\frac{y}{1-y} = C' e^{rt}$$

Impose $y(0)=\frac{1}{10}$:

$$\frac{\frac{1}{10}}{1-\frac{1}{10}} = C\cancel{e^{r0}}^{\,1} \qquad C = \frac{\frac{1}{10}}{\frac{9}{10}} = \frac{1}{9}$$

Impose $y(1)=\frac{2}{10}$:

$$\frac{\frac{2}{10}}{1-\frac{2}{10}} = \frac{1}{9}e^{r\cdot 1} \Rightarrow \frac{\frac{2}{10}}{\frac{8}{10}} = \frac{1}{9}e^{r}$$

$$\frac{1}{9}e^{r} = \frac{1}{4} \Rightarrow e^{r} = \frac{9}{4}.$$

So $$\boxed{\frac{y}{1-y} = \frac{1}{9}(e^{r})^{t} = \frac{1}{9}\left(\frac{9}{4}\right)^{t}}$$

$$y = \frac{1}{9}\left(\frac{9}{4}\right)^{t}[1-y]$$

$$y + \frac{1}{9}\left(\frac{9}{4}\right)^{t} y = \frac{1}{9}\left(\frac{9}{4}\right)^{t}$$

$$\boxed{y = \frac{\frac{1}{9}\left(\frac{9}{4}\right)^{t}}{1+\frac{1}{9}\left(\frac{9}{4}\right)^{t}} = \frac{\left(\frac{9}{4}\right)^{t}}{9+\left(\frac{9}{4}\right)^{t}}}$$

(a) $$y(2) = \frac{\left(\frac{9}{4}\right)^{2}}{9+\left(\frac{9}{4}\right)^{2}} = \frac{9}{25} = 0.36$$

So after 2 years, the ants are at $\boxed{36\%}$ of maximum capacity

(b) To solve (b), it is easier to use

$$\frac{y}{1-y} = \frac{1}{9}\left(\frac{9}{4}\right)^t.$$

Impose $y = \frac{9}{10}$ and solve for t:

$$\frac{\frac{9}{10}}{1-\frac{9}{10}} = \frac{1}{9}\left(\frac{9}{4}\right)^t$$

$$\frac{\frac{9}{10}}{\frac{1}{10}} = \frac{1}{9}\left(\frac{9}{4}\right)^t$$

$$9 = \frac{1}{9}\left(\frac{9}{4}\right)^t$$

$$81 = \left(\frac{9}{4}\right)^t$$

$$t = \frac{\ln(81)}{\ln\left(\frac{9}{4}\right)} \approx \boxed{5.42 \text{ years.}}$$

(c): Population growing fastest when y' is maximized, or

when $y'' = 0$.

$$y'' = \frac{d}{dt}\left[\frac{dy}{dt}\right] = \frac{d}{dt}\left[r(1-y)y\right]$$

$$= r\left[-y + (1-y)\right]$$

$$= r\left[1-2y\right]$$

Set $y'' = 0$:

$$0 = r(1-2y)$$

$$0 = 1-2y$$

$$2y = 1$$

$$y = \frac{1}{2}$$

Find corresponding time t:

$$\frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{1}{9}\left(\frac{9}{4}\right)^t$$

$$1 = \frac{1}{9}\left(\frac{9}{4}\right)^t$$

$$9 = \left(\frac{9}{4}\right)^t$$

$$\ln 9 = t \ln\left(\frac{9}{4}\right)$$

$$t = \frac{\ln(9)}{\ln\left(\frac{9}{4}\right)} \approx \boxed{2.71 \text{ years}}$$

2. Find an integrating factor $\mu(x)$ that depends only on x to solve

$$\frac{dy}{dx} = -\left(\frac{y\sin x + 2yx(\cos x)}{x\sin x}\right).$$

Hint: rewrite the equation in standard differential form. Try imposing $\psi_y = N$ first.

① $(x\sin x)dy = -(y\sin x + 2yx\cos x)dx$

$\underbrace{(y\sin x + 2yx\cos x)}_{M}dx + \underbrace{(x\sin x)}_{N}dy = 0$

- $M_y = \sin x + 2x\cos x$
- $N_x = \sin x + x\cos x$

Not equal, not exact.

- $\frac{M_y - N_x}{N} = \frac{x\cos x}{x\sin x} = \cancel{\frac{\cos x}{\sin x}}\ \cot x$ ← only depends on x
- $\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu$
- $\frac{d\mu}{dx} = \cot x\,\mu$
- $\int \frac{1}{\mu}d\mu = \int \cot x\,dx$

$\ln|\mu| = \ln|\sin x| + C$

② $\mu = C'\sin x$

Choose $C' = 1$.

So New Eqn is:

$\underbrace{y\sin^2 x + 2yx\cos x\sin x}_{\text{new } M}dx + \underbrace{x\sin^2 x}_{\text{New } N}dy = 0$

③ Impose $\psi_x = M$.

$\psi = \int y\sin^2 x + \ldots\, dx$. Messy!

Try to impose $\psi_y = N$ first:

$\psi = \int x\sin^2 x\,dy = (x\sin^2 x)y + h(x)$

Now Impose $\psi_x = M$

$\frac{\partial}{\partial x}\left[(x\sin^2 x)y + h(x)\right] = y\sin^2 x + 2yx(\cos x\sin x)$

$\cancel{y\sin^2 x} + \cancel{yx\cdot 2\sin x\cos x} + h'(x) = \cancel{y\sin^2 x} + \cancel{2yx(\cos x)(\sin x)}$

$h'(x) = 0$

$h(x) = C$

$\psi = (x\sin^2 x)y + C$

$\boxed{xy\sin^2 x = C}$

3. Compute the following.

(a) $\frac{3+2i}{4-i}$

$\frac{3+2i}{4-i}\cdot\frac{(4+i)}{(4+i)} = \frac{12 + 8i + 3i + 2i^2}{16 - 4i + 4i - i^2}$

$= \frac{12 + 11i - 2}{16 - (-1)} = \frac{10 + 11i}{17}$

$= \boxed{\frac{10}{17} + \frac{11}{17}i}$

(b) $(2+i)e^{1-\frac{\pi}{4}i}$

$= (2+i)e^1\cdot e^{-\frac{\pi}{2}i}$

$= (2+i)e\cdot\left(\cos(-\tfrac{\pi}{4}) + i\sin(-\tfrac{\pi}{4})\right)$

$= (2+i)e\cdot\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$

$= (2+i)e\frac{\sqrt{2}}{2}(1 - i)$

$= \frac{e\sqrt{2}}{2}\left[(2+i)(1-i)\right] = \frac{e\sqrt{2}}{2}\left[2 + i - 2i - i^2\right]$

$= \frac{e\sqrt{2}}{2}\left[3 - i\right] = \boxed{\frac{3e\sqrt{2}}{2} - \frac{e\sqrt{2}}{2}i}$