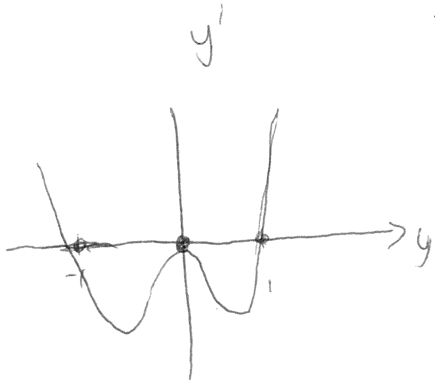
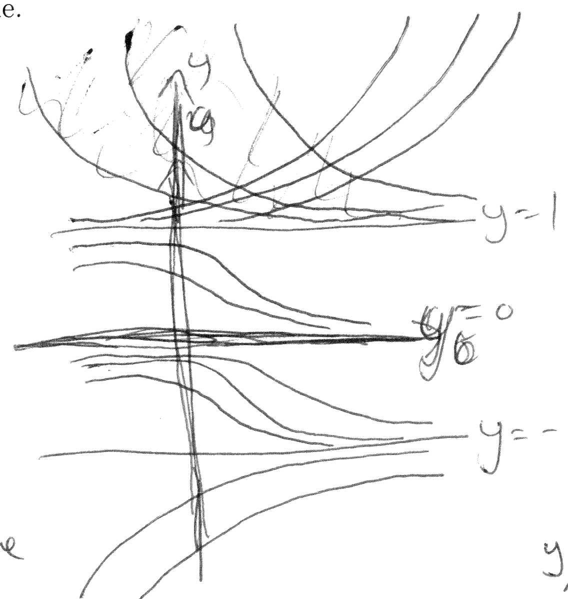
Solutions

1. Give qualitative analysis of the following autonomous differential equations. That is, determine the equilibrium solutions, classify each as stable, unstable, or semistable, and sketch the solutions. Include a phase line.

   (a) $\frac{dy}{dt} = y^2(y^2 - 1)$



phase line



unstable

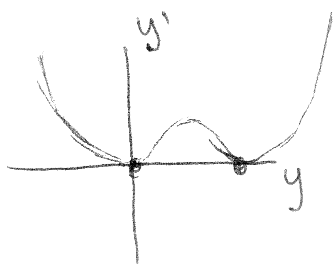
semistable

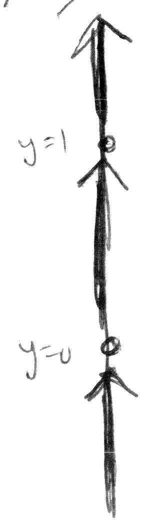
stable

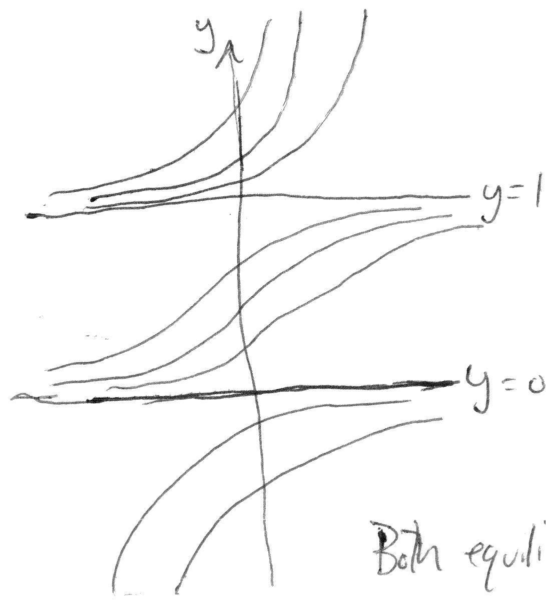
   (b) $\frac{dy}{dt} = y^2(1 - y)^2$

$y^2(1-y)^2 = 0$

$y = 0$ or $y = 1$





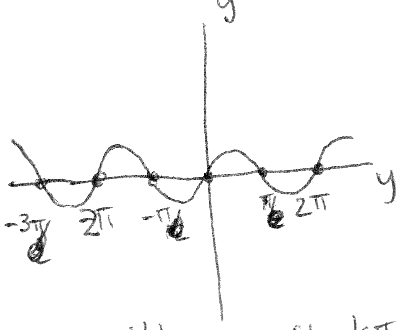


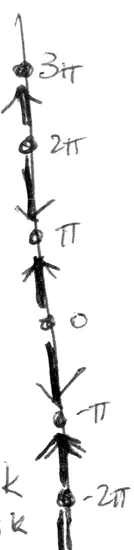
Both equilibrium solns are semistable

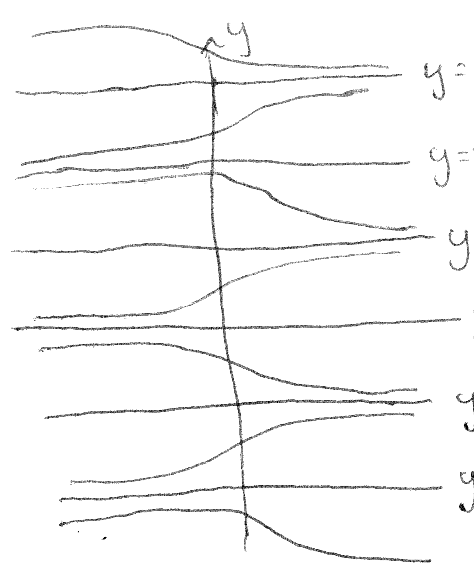
   (c) $\frac{dy}{dt} = \sin y$

$\sin y = 0$

$y = k \cdot \pi$ for some integer k







	Stable?
$y=3\pi$	yes
$y=2\pi$	no
$y=\pi$	yes
$y=0$	no
$y=-\pi$	yes
$y=-2\pi$	no

Stable equilibria: $y = k\pi$ for odd integers k

unstable equilibria: $y = k\pi$ for even integers k

2. Determine whether the following equations are exact. If exact, find the solution.

(a) $(2x+3)+(2y-2)y'=0$

~~Require $\psi_y = 2y-2$: $\psi = \int 2y-2\,dy = y^2-2y$~~

⓪ Check $M_y \overset{?}{=} N_x$

$0 = 0$ ✓ $\boxed{\text{Exact}}$

① Require $\psi_x = M$:

$\psi_x = 2x+3$

$\int \psi_x \, dx = \int 2x+3 \, dx$

$\psi = x^2+3x+h(y)$

② Require $\psi_y = N$

$\psi_y = 2y-2$

$\frac{\partial}{\partial y}\left[x^2+3x+h(y)\right] = 2y-2$

$h'(y) = 2y-2$

$h(y) = \int 2y-2\,dy$

$= y^2-2y+C$

$\psi = x^2+3x+y^2$ ...

$\boxed{x^2+3x+y^2-2y+ \ldots}$

(b) $(2x+4y)+(2x-2y)y'=0$

~~① Require $\psi_x = M$: $\psi = \int (2x+4y)\,dx$~~ Impose

⓪ Check $M_y \overset{?}{=} N_x$

$4 \overset{?}{=} 2$ No, so this equation is $\boxed{\text{not exact}}$.

(c) $(2xy^2+2y)+(2x^2y+2x)y'=0$

⓪ Check $M_y \overset{?}{=} N_x$

$\frac{\partial}{\partial y}\left[2xy^2+2y\right] \overset{?}{=} \frac{\partial}{\partial x}\left[2x^2y+2x\right]$

$4xy+2 \overset{?}{=} 4xy+2$ ✓

$\boxed{\text{Exact}}$

① Require $\psi_x = M$:

$\psi = \int 2xy^2+2y\,dx$

$= x^2y^2+2yx+h(y)$

② Require $\psi_y = N$:

$\frac{\partial}{\partial y}\left[x^2y^2+2yx+h(y)\right]$

$= 2x^2y+2 \ldots$

$\cancel{2x^2y+2x}+h'$

$= \cancel{2x^2y+2x}$

$h'(y) = 0$

$h(y) = C$

so $\psi = 0$:

$\boxed{x^2y^2+2yx+C=0}$

(d) $y' = -\frac{ax+by}{bx+cy}$ where a, b, and c are constants.

$(bx+cy)y' = -ax+by$

$\underbrace{(ax+by)}_{M} + \underbrace{(bx+cy)}_{N}y' = 0$

⓪ Check $M_y \overset{?}{=} N_x$:

$0+b \overset{?}{=} b+0$

$\boxed{\text{Exact}}$

① Impose $\psi_x = M$

$\psi = \int (ax+by)\,dx$

$= \frac{a}{2}x^2+byx+h(y)$

② Impose $\psi_y = N$

$\frac{\partial}{\partial y}\left[\frac{a}{2}x^2+byx+h(y)\right] = N$

$\cancel{bx}+h'(y) = \cancel{bx}+cy$

$h(y) = \int cy\,dy = \frac{c}{2}y^2$.

So $\psi = 0$:

$\boxed{\frac{a}{2}x^2+byx+\frac{c}{2}y^2+k=0}$