Solutions. (1)

1. Find the general solution to the following.

   (a) $\mathbf{x}' = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \mathbf{x}$

   (b) [7.6.7] $\mathbf{x}' = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} \mathbf{x}$

   (c) $\mathbf{x}' = \begin{bmatrix} -4 & -9 & 3 \\ 0 & -1 & 0 \\ -6 & -18 & 5 \end{bmatrix} \mathbf{x}$

(a)

① $\begin{vmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0$

$(3-\lambda)(1-\lambda) + 1 = 0$

$\lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2)^2 = 0$; $\lambda = 2$ with mult. 2.

② $\underline{\lambda = 2}$:

$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad x_1 - x_2 = 0$

$\xi = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$

No ~~other~~ other independent ~~such~~ eigenvector. Try

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{2t} + \eta e^{2t}$ where $(A - 2I)\eta = \xi$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \eta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & -1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right] \qquad x_1 - x_2 = 1$$

So $\eta = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ or, better yet, $\eta = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We get a second soln: $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{2t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t}$

3 Gen soln:

$$\boxed{x = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{2t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} \right)}$$

(b)

1. Find eigenvalues:

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^3 - [-4(1-\lambda)] = 0$$

$$(1-\lambda)[(1-\lambda)^2 + 4] = 0$$

$$(1-\lambda)[\lambda^2 - 2\lambda + 5] = 0$$

$\lambda = 1$ or $\lambda^2 - 2\lambda + 5 = 0$

$$\lambda = \frac{2 \pm \sqrt{4 - 4\cdot 5}}{2}$$

$$= 1 \pm \sqrt{1-5}$$

$$= 1 \pm \sqrt{-4}$$

$$= 1 \pm 2i$$

2. Find eigenvectors and associated solns

$\lambda = 1$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 - x_3 = 0$

$2x_2 + 3x_3 = 0$

Try $x_3 = +2$; $\xi = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$; $x = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} e^t$

$\lambda = \overset{1-2i}{\cancel{1+2i}}$, $\lambda = 1+2i$ (Complex Conjugate Pair)

$$\begin{bmatrix} 2i & 0 & 0 \\ 2 & 2i & -2 \\ 3 & 2 & 2i \end{bmatrix} \underset{\sim}{R_1 \leftarrow \frac{1}{2i} \cdot R_1} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2i & -2 \\ 3 & 2 & 2i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & -1 \\ 0 & 1 & i \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & i \\ 0 & i & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & i \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0$

$x_2 + i x_3 = 0$ Try $x_3 = i$.

$$\xi = \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix}$$

$$x = \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix} e^{(1-2i)t}$$

(Also, $x = \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix} e^{(1+2i)t}$ but we need only 1 of these.)

To get real valued solns, extract real & imaginary parts:

$$x = \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix} e^t \cdot e^{-2ti}$$

$$= \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix} e^t \left(\cos(-2t) + i \sin(-2t) \right)$$

$$= \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix} e^t \left(\cos(2t) - i \sin(2t) \right)$$

$$= \begin{bmatrix} 0 \\ 1(\cos(2t) - i\sin(2t)) \\ i(\cos(2t) - i\sin(2t)) \end{bmatrix} e^t$$

$$= \begin{bmatrix} 0 \\ \cos(2t) - i\sin(2t) \\ \sin(2t) + i\cos(2t) \end{bmatrix} e^t$$

$$= \left(\begin{bmatrix} 0 \\ \cos(2t) \\ \sin(2t) \end{bmatrix} + \begin{bmatrix} 0 \\ -\sin(2t) \\ \cos(2t) \end{bmatrix} i \right) e^t$$

$$= \underbrace{\begin{bmatrix} 0 \\ \cos(2t) \\ \sin(2t) \end{bmatrix} e^t}_{\text{Real Part}} + \underbrace{\begin{bmatrix} 0 \\ -\sin(2t) \\ \cos(2t) \end{bmatrix}}_{\text{Imaginary Part}} e^t \cdot i$$

3 General soln:

$$\boxed{X = C_1 \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} e^t + C_2 \begin{bmatrix} 0 \\ \cos(2t) \\ \sin(2t) \end{bmatrix} e^t + C_3 \begin{bmatrix} 0 \\ -\sin(2t) \\ \cos(2t) \end{bmatrix} e^t}$$

(c) 1 Find Eigen~~vector~~values:

$$\begin{vmatrix} -4-\lambda & -9 & 3 \\ 0 & -1-\lambda & 0 \\ -6 & -18 & 5-\lambda \end{vmatrix} = 0$$

$$(-4-\lambda)(-1-\lambda)(5-\lambda) - \left[-18(-1-\lambda)\right] = 0$$

$$(-1-\lambda)\left[(-4-\lambda)(5-\lambda) + 18\right] = 0$$

$$(\lambda+1)\left[(-4-\lambda)(5-\lambda) + 18\right] = 0$$

$$(\lambda+1)\left[\lambda^2 - \lambda - 2\right] = 0$$

$$(\lambda+1)(\lambda-2)(\lambda+1) = 0$$

$\lambda = 2$, $\lambda = -1$ w/ mult. 2.

2 Find eigenvectors and corresponding solns:

$\underline{\lambda=2}$:

$$\begin{bmatrix} -6 & -9 & 3 \\ 0 & -3 & 0 \\ -6 & -18 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 6 & 9 & -3 \\ 0 & 1 & 0 \\ 0 & -9 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 6 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$2x_1 - x_3 = 0$

$x_2 = 0$

$$\xi = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}; \quad x = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} e^{2t}$$

$\underline{\lambda = -1}$:

$$\begin{bmatrix} -3 & -9 & 3 \\ 0 & 0 & 0 \\ -6 & -18 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 \\ -1 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + 3x_2 - x_3 = 0$$

Two free variables, so we get 2 independent eigenvectors.

Try $\left.\begin{matrix} x_2 = 1 \\ x_3 = 0 \end{matrix}\right\}$ $\xi = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$

$$x = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} e^{-t}$$

Try $\begin{matrix} x_2 = 0 \\ x_3 = 1 \end{matrix}$ $\xi' = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$x = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{-t}$$

3 The general soln is

$$\boxed{X = c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{-t}}$$