Solutions

1. Let $A = \begin{bmatrix} -5 & -3 & -15 \\ 1 & 0 & 2 \\ 2 & 1 & 6 \end{bmatrix}$.

   (a) Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} -5 & -3 & -15 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & 6 & 0 & 0 & 1 \end{array}\right]$$

Adj by R_2 $\rightsquigarrow \left[\begin{array}{ccc|ccc} 0 & -3 & -5 & 1 & 5 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & -2 & 1 \end{array}\right]$

Swap $\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & -2 & 1 \\ 0 & -3 & -5 & 1 & 5 & 0 \end{array}\right]$

$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & -2 & 1 \\ 0 & 0 & 1 & 1 & -1 & 3 \end{array}\right]$

$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -6 \\ 0 & 1 & 0 & -2 & 0 & -5 \\ 0 & 0 & 1 & 1 & -1 & 3 \end{array}\right]$

$$\boxed{A^{-1} = \begin{bmatrix} -2 & 3 & -6 \\ -2 & 0 & -5 \\ 1 & -1 & 3 \end{bmatrix}}$$

   (b) Use (a) to solve the system

$$\begin{array}{rcrcrcl} -5x_1 & - & 3x_2 & - & 15x_3 & = & 3 \\ x_1 & + & 0x_2 & + & 2x_3 & = & 2 \\ 2x_1 & + & x_2 & + & 6x_1 & = & 1 \end{array}.$$

$$Ax = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\cancel{A^{-1}A}\, x = A^{-1}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$x = A^{-1}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = 3\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + 1\begin{bmatrix} -6 \\ -5 \\ 3 \end{bmatrix} = \boxed{\begin{bmatrix} -6 \\ -11 \\ 4 \end{bmatrix}}$$

2. Find all solutions to

$$\begin{array}{rcrcrcl} x_1 & - & x_2 & + & x_3 & = & 2 \\ -2x_1 & + & 6x_2 & - & 3x_3 & = & 5 \end{array}.$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ -2 & 6 & -3 & 5 \end{array}\right]$$

$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 4 & -1 & 9 \end{array}\right]$

$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -\frac{1}{4} & \frac{9}{4} \end{array}\right]$

$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{4} & \frac{17}{4} \\ 0 & 1 & -\frac{1}{4} & \frac{9}{4} \end{array}\right]$

Let $x_3 = c$, ~~for some const~~ c

$$\left.\begin{array}{l} x_1 + \frac{3}{4}\cancel{x_3} = \frac{17}{4} \\ x_2 - \frac{1}{4}\cancel{x_3} = \frac{9}{4} \\ x_3 = c \end{array}\right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{17}{4} \\ \frac{9}{4} \\ 0 \end{bmatrix} + c\begin{bmatrix} -\frac{3}{4} \\ \frac{1}{4} \\ 1 \end{bmatrix},$$

where c is arbitrary

3. Find all eigenvector/eigenvalue pairs for the following matrices.

(a) $\begin{bmatrix} 7 & 8 \\ -4 & -5 \end{bmatrix}$.

① $0 = \begin{vmatrix} 7-\lambda & 8 \\ -4 & -5-\lambda \end{vmatrix} = (7-\lambda)(-5-\lambda) + 32$

$0 = (\lambda-7)(\lambda+5) + 32$

$0 = \lambda^2 - 2\lambda - 35 + 32$

$0 = \lambda^2 - 2\lambda - 3$

$0 = (\lambda - 3)(\lambda+1)$

② $\lambda = 3$: $\left[\begin{array}{cc|c} 4 & 8 & 0 \\ -4 & -8 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 4 & 8 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\xi_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

③ $\lambda = -1$: $\left[\begin{array}{cc|c} 8 & 8 & 0 \\ -4 & -4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\xi_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

④ So $\boxed{\begin{array}{ll} \lambda_1 = 3, & \xi_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ \lambda_2 = -1 & \xi_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{array}}$

(b) $\begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix}$.

① $0 = \begin{vmatrix} 4-\lambda & -3 \\ 6 & -2-\lambda \end{vmatrix} = (4-\lambda)(-2-\lambda) + 18$

$0 = (\lambda-4)(\lambda+2) + 18$

$0 = \lambda^2 - 2\lambda + 10$

$\lambda = \frac{2 \pm \sqrt{4 - 4\cdot 10}}{2} = 1 \pm \sqrt{-3} = 1 \pm 3i$

② $\underline{\lambda = 1+3i}$:

$\left[\begin{array}{cc|c} 4-(1+3i) & -3 & 0 \\ 6 & -2-(1+3i) & 0 \end{array}\right]$

$\to \left[\begin{array}{cc|c} 3-3i & -3 & 0 \\ 6 & -3-3i & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 1-i & -1 & 0 \\ -2 & 1+i & 0 \end{array}\right]$

Mult R_1 by $1+i$ $\to$ $\left[\begin{array}{cc|c} 2 & -(1+i) & 0 \\ -2 & 1+i & 0 \end{array}\right]$

$\to \left[\begin{array}{cc|c} 2 & -(1+i) & 0 \\ 0 & 0 & 0 \end{array}\right]$

$2x_1 - (1+i)x_2 = 0$

~~$2\xi_1 - (1+i)\xi_2 = 0$~~

$\xi_1 = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$.

③ $\lambda_2 = \overline{\lambda_1} = 1-3i$

$\xi_2 = \overline{\xi_1} = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$.

(see next page) → (c) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ -4 & 4 & 3 \end{bmatrix}$ ②

3 b, cont.

So $\lambda_1 = 1+3i$, $\xi_1 = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$

$\lambda_2 = 1-3i$, $\xi_2 = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$

3(c)

$$0 = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 0 & -1-\lambda & 0 \\ -4 & 4 & 3-\lambda \end{vmatrix}$$

$$= (1-\lambda)(-1-\lambda)(3-\lambda) + 0 + 0 - 0 - 0 - 0$$

$$= [-(\lambda-1)][-(\lambda+1)][-(\lambda-3)]$$

$$= -(\lambda-1)(\lambda+1)(\lambda-3)$$

$\lambda=1$, $\lambda=-1$, $\lambda=3$.

$\lambda=1$: $\begin{bmatrix} 0 & 2 & 0 \\ 0 & -2 & 0 \\ -4 & 4 & 2 \end{bmatrix}$

$\rightsquigarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -4 & 0 & 2 \end{bmatrix}$

$\rightsquigarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 \quad -\frac{1}{2}x_3 = 0$

$x_2 = 0$

Let $x_3 = c$. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}c \\ 0 \\ c \end{bmatrix}$; $c=2 \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

$\underline{\lambda = -1:}$

$$\begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 0 \\ -4 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 + x_2 = 0$

$x_3 = 0$

Let $x_2 = c$. Then $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -c \\ c \\ 0 \end{bmatrix}$; $\xrightarrow{c=-1} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

$\underline{\lambda = 3:}$

$$\begin{bmatrix} -2 & 2 & 0 \\ 0 & -4 & 0 \\ -4 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0$

$x_2 = 0$.

Let $x_3 = c$. We get $\begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$. Try $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$\underline{\text{So}}$: $\boxed{\begin{array}{ll} \lambda_1 = 1, \ \xi_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} & \lambda_3 = 3 \quad \xi_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ \lambda_2 = -1, \ \xi_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} & \end{array}}$