Solutions



1. Compute the following.

(a) $\mathcal{L}\{f(t)\}$ where $f(t) = \begin{cases} 0 & \text{if } t < 2 \\ t^2 & \text{if } 2 \le t \end{cases}$ $\quad f(t) = u_2(t)\,t^2 = u_2(t)\left[(t-2)^2 + 4t - 4\right]$

$f(t) = u_2(t)\left[(t-2)^2 + (4(t-2)+8) - 4\right] = u_2(t)\left((t-2)^2 + 4(t-2) + 4\right)$

$\mathcal{L}\{f(t)\} = e^{-2s}\left(\mathcal{L}\{t^2+4t+4\}\right) = \boxed{e^{-2s}\left(\frac{2!}{s^3} + 4\cdot\frac{1}{s^2} + 4\cdot\frac{1}{s}\right)}$

(b) $\mathcal{L}\{g(t)\}$ where $g(t) = \begin{cases} 3 & \text{if } t < \pi \\ \cos t & \text{if } t \ge \pi \end{cases}$. $= (1-u_\pi(t))\cdot 3 + u_\pi(t)\cos(t)$

$= 3 - u_\pi(t)\cdot 3 + u_\pi(t)\cos((t-\pi)+\pi) = 3 - u_\pi(t)\cdot 3 - u_\pi(t)\cos(t-\pi)$

$= 3 - u_\pi(t)(3+\cos(t-\pi))$

$\mathcal{L}\{g(t)\} = \mathcal{L}\{3\} - \mathcal{L}\{u_\pi(t)(3+\cos(t-\pi))\} = \frac{3}{s} - e^{-\pi s}\mathcal{L}\{3+\cos t\}$

$= \boxed{\frac{3}{s} - e^{-\pi s}\left(\frac{3}{s} + \frac{s}{s^2+1}\right)}$

(c) $\mathcal{L}^{-1}\left\{\frac{1-e^{-3s}}{s^2+6s+10}\right\} = \mathcal{L}^{-1}\left\{\frac{1-e^{-3s}}{(s+3)^2+1}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s+3)^2+1}\right\}$

$= e^{-3t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - u_3(t)\,\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+1}\right\}\Big|_{(t-3)} = e^{-3t}\sin(t) - u_3(t)\left(e^{-3t}\sin(t)\right)\Big|_{t-3}$

$= \boxed{e^{-3t}\sin(t) - u_3(t)\,e^{-3(t-3)}\sin(t-3)}$

(d) $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)(s^2+6s+10)}\right\}$

$\frac{s}{(s^2+1)(s^2+6s+10)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+6s+10}$

$(As+B)(s^2+6s+10) + (Cs+D)(s^2+1) = s$

$s=i$: $(Ai+B)(-1+6i+10) + 0 = i$

$(B+Ai)(9+6i) = 0+i$

$(9B-6A) + (9A+6B)i = 0+i$

$\left[-6A+9B = 0\right]\cdot\frac{3}{2}$

$9A+6B = 1$

$\frac{39}{2}B = 1$ ; $B = \frac{2}{39}$

$A = \frac{9}{6}B = \frac{3}{2}\cdot\frac{2}{39} = \frac{3}{39}$

See pg. 3

1(d) continued:

$$(As+B)(s^2+6s+10)+(Cs+1)(s^2+1)=s; \quad A=\frac{3}{39}, \quad B=\frac{2}{39}.$$

$$(A+C)s^3+\underbrace{(6A+B+D)}s^2+\underbrace{(10A+6B+C)}s+(10B+D)=s$$

not needed

$s^3$: $A+C=0$

$$C=-A=-\frac{3}{39}$$

$s^0$: $10B+D=0;\quad D=-10B=\frac{-20}{39}.$

So:

$$A=\frac{3}{39} \qquad C=\frac{-3}{39}$$

$$B=\frac{2}{39} \qquad D=\frac{-20}{39}$$

$$\frac{s}{(s^2+1)(s^2+6s+10)}=\frac{\frac{3}{39}s+\frac{2}{39}}{s^2+1}+\frac{-\frac{3}{39}s-\frac{20}{39}}{(s+3)^2+1}$$

$$=\frac{1}{39}\left(3\cdot\frac{s}{s^2+1}+2\cdot\frac{1}{s^2+1}-\frac{3s+20}{(s+3)^2+1}\right)$$

$$=\frac{1}{39}\left(3\cdot\frac{s}{s^2+1}+2\cdot\frac{1}{s^2+1}-\frac{3(s+3)+11}{(s+3)^2+1}\right)$$

$$=\frac{1}{39}\left(3\cdot\frac{s}{s^2+1}+2\cdot\frac{1}{s^2+1}-3\,\frac{s+3}{(s+3)^2+1}-11\cdot\frac{1}{(s+3)^2+1}\right)$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)(s^2+6s+10)}\right\}=\boxed{\frac{1}{39}\left(3\cos(t)+2\sin(t)-3e^{-3t}\cos(t)-11e^{-3t}\sin(t)\right)}$$

2. [6.4.13] Solve $y^{(4)} + 5y'' + 4y = 1 - u_\pi(t)$ with $y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0$.

$$\mathcal{L}\{y^{(4)}\} + 5\mathcal{L}\{y^{(2)}\} + 4\mathcal{L}\{y\} = \mathcal{L}\{1 - u_\pi(t)\}$$

$$\left(s^4 Y \cancel{- s^3 y(0) - s^2 y'(0) - s y''(0) - y^{(3)}(0)}\right) + 5\left(s^2 Y \cancel{- s y(0) - y'(0)}\right) + 4Y = \mathcal{L}\{1 - u_\pi(t)\}$$

$$(s^4 + 5s^2 + 4)Y = \mathcal{L}\{1\} - \mathcal{L}\{u_\pi(t)\}$$

$$(s^4 + 5s^2 + 4)Y = \frac{1}{s} - \frac{e^{-\pi s}}{s}$$

$$Y = \frac{1 - e^{-\pi s}}{s(s^4 + 5s^2 + 4)} = \frac{1 - e^{-\pi s}}{s(s^2+4)(s^2+1)} = H(s)\left(1 - e^{-\pi s}\right), \text{ where } H(s) = \frac{1}{s(s^2+4)(s^2+1)}$$

$$y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\{H(s) - e^{-\pi s} H(s)\} = h(t) - u_\pi(t) h(t - \pi)$$

$$h(t) = \mathcal{L}^{-1}\{H(t)\}$$

$$\frac{1}{s(s^2+4)(s^2+1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4} + \frac{Ds + E}{s^2 + 1}$$

$$A(s^2+4)(s^2+1) + (Bs + C)s(s^2+1) + (Ds + E)s(s^2+4) = 1$$

$\underline{s=0}$: $A \cdot 4 = 1$; $A = \frac{1}{4}$

$\underline{s=i}$: $0 + (Bi + C)i(\cancel{i^2 + 1}^{\,0}) + (Di + E)i(i^2 + 4) = 1$

$$\left.\begin{aligned} (Di + E)i \cdot 3 &= 1 \\ -3D + 3Ei &= 1 \end{aligned}\right\} \quad \left.\begin{aligned} -3D &= 1 \\ 3E &= 0. \end{aligned}\right\} \quad \begin{aligned} D &= -\tfrac{1}{3} \\ E &= 0. \end{aligned}$$

Cont. on p. 4

$s = \underline{2i}$:

$$A\cancel{(0)}(s^2+1)^{\nearrow 0} + (B(2i)+C)(2i)((2i)^2+1) + \cdots^{\nearrow 0} = 1$$

$$(C + 2iB)(2i)(-4+1) = 1$$

$$(C + 2iB)(-6i) = 1$$

$$12B - 6Ci = 1$$

$$\left.\begin{aligned} 12B &= 1 \\ -6C &= 0 \end{aligned}\right\} \quad \begin{aligned} B &= \tfrac{1}{12} \\ C &= 0. \end{aligned}$$

$$\frac{1}{s(s^2+4)(s^2+1)} = \frac{1/4}{s} + \frac{\frac{1}{12}s + 0}{s^2+4} + \frac{-\frac{1}{3}s + 0}{s^2+1}$$

$$= \frac{1}{4}\cdot\frac{1}{s} + \frac{1}{12}\cdot\frac{s}{s^2+2^2} - \frac{1}{3}\,\frac{s}{s^2+1}$$

$$\boxed{\begin{aligned} h(t) &= \mathcal{L}^{-1}\{H(s)\} = \frac{1}{4}\cdot 1 + \frac{1}{12}\cos(2t) - \frac{1}{3}\cos(t) \\ y(t) &= h(t) - u_\pi(t)\, h(t-\pi). \end{aligned}}$$