Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[24 points]** A mass of 1 kg stretches a spring by 5 cm. The spring/mass system is enclosed in a medium which imparts a viscous force of magnitude 24 N when the mass moves at a velocity of 2 m/s. An external motor imparts a force of $2\cos(5t)$ (in N). Solve for the forced response $U(t)$, expressing $U(t)$ in the form $R\cos(\omega t - \delta)$. Hint: be careful with units.

(in m)

and approximating to 5 decimal places

① Units: s, m, kg, N.

kx ↑, ↓ mg

$mg = kx$

$1\ (kg)\cdot(9.8)\ m/s^2 = k\cdot\frac{5}{100}\ m$

$k = 196\ N/m.$

$F_d = \gamma u'$

$24N = \gamma\cdot 2\ \frac{m}{s}$ $\Big\} \Rightarrow \gamma = 12\ \frac{N}{m/s}$

$mu'' + \gamma u' + ku = 2\cos 5t$

$u'' + 12u' + 196u = 2\cos(5t)$

② $[U(t) = A\sin(5t) + B\cos(5t)]\cdot 196$

$[U' = -5B\sin(5t) + 5A\cos(5t)]\cdot 12$

$[U'' = -25A\sin(5t) - 25B\cos(5t)]\cdot 1$

$(-25A - 60B + 196A)\sin(5t) + (-25B + 60A + 196B)\cos(5t) = 2\cos 5t$

$171A - 60B = 0$

$60A + 171B = 2$

$\begin{vmatrix} 171 & -60 \\ 60 & 171 \end{vmatrix} = (171)^2 - (-60)(60) = 32{,}841$

$A = \dfrac{\begin{vmatrix} 0 & -60 \\ 2 & 171 \end{vmatrix}}{\begin{vmatrix} 171 & -60 \\ 60 & 171 \end{vmatrix}} = \dfrac{120}{32841} = \dfrac{40}{10947} \approx 0.00365$

$B = \dfrac{\begin{vmatrix} 171 & 0 \\ 60 & 2 \end{vmatrix}}{\begin{vmatrix} 171 & -60 \\ 60 & 171 \end{vmatrix}} = \dfrac{342}{32841} = \dfrac{38}{3649} \approx 0.01041$

$U(t) = A\sin(5t) + B\cos(5t)$

③ $U(t) = \dfrac{40}{10947}\sin(5t) + \dfrac{38}{3649}\cos(5t)$

$R = \sqrt{A^2 + B^2} = \left(\dfrac{4}{32841}\right)^{1/2}$

≈ 0.01104

$\delta = \tan^{-1}\left(\dfrac{A}{B}\right) \approx 0.33746$

$\boxed{U(t) \approx (0.01104)\cos(5t - 0.33746)}$

2. [**4 parts, 8 points each**] Compute the following.

(a) $\mathcal{L}\{t(t+1)\}$

$\mathcal{L}\{t^2+t\}$

$= \mathcal{L}\{t^2\} + \mathcal{L}\{t\}$

$= \boxed{\frac{2}{s^3} + \frac{1}{s^2}}$

(b) $\mathcal{L}\{2\cosh(3t) - u_5(t)(t+1)\}$

$= 2\mathcal{L}\{\cosh(3t)\} - \mathcal{L}\{u_5(t)((t-5)+6)\}$

$= 2 \cdot \frac{s}{s^2-9} - e^{-5s}\mathcal{L}\{t+6\}$

$= \boxed{\frac{2s}{s^2-9} - e^{-5s}\left(\frac{1}{s^2} + \frac{6}{s}\right)}$

(c) $\mathcal{L}^{-1}\left\{\frac{6}{(s-7)^5}\right\} = e^{7t}\mathcal{L}^{-1}\left\{\frac{6}{s^5}\right\}$

$= e^{7t} \cdot \frac{6}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{s^5}\right\}$

$= e^{7t} \cdot \frac{\cancel{6}}{4 \cdot \cancel{3} \cdot \cancel{2} \cdot 1} \cdot t^4$

$= e^{7t} \cdot \frac{1}{4} \cdot t^4$

$= \boxed{\frac{t^4 e^{7t}}{4}}$

(d) $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+4)(s+2)}\right\}$

$\frac{s^2}{(s^2+4)(s+2)} = \frac{As+B}{s^2+4} + \frac{C}{s+2}$

$(s+2)(As+B) + (s^2+4)C = s^2 \quad (*)$

$(A+C)s^2 + (2A+B)s + 2B+4C = s^2$

In $(*)$, $s=-2$:

$0 + 8C = 4; \; C = \frac{1}{2}$

$A+C=1; \; A = 1-C = 1-\frac{1}{2} = \frac{1}{2}$

$2A+B=0; \; B=-2A=-1$

$\frac{s^2}{(s^2+4)(s+2)} = \frac{\frac{1}{2}s - 1}{s^2+4} + \frac{1}{2}\frac{1}{s+2}$

$= \frac{1}{2}\frac{s}{s^2+4} - \frac{1}{2}\frac{2}{s^2+4} + \frac{1}{2} \cdot \frac{1}{s+2}$

(d), continued:

$\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$

$= \boxed{\frac{1}{2}\cos(2t) - \frac{1}{2}\sin(2t) + \frac{1}{2}e^{-2t}}$

3. **[20 points]** Use the Laplace transform to solve: $y'' - 3y' - 10y = e^t$, $y(0) = 0$ and $y'(0) = 1$.

①

$$\left[s^2Y - sy(0) - y'(0)\right] - 3\left[sY - y(0)\right] - 10Y = \mathcal{L}\{e^t\}$$

$$\left[s^2Y - 1\right] - 3\left[sY\right] - 10Y = \frac{1}{s-1}$$

$$(s^2 - 3s - 10)Y = \frac{1}{s-1} + 1$$

$$Y = \frac{1}{(s-1)(s^2-3s-10)} + \frac{(s-1)}{(s^2-3s-10)(s-1)}$$

$$= \frac{s}{(s-1)(s^2-3s-10)} = \frac{s}{(s-1)(s-5)(s+2)}$$

$$\frac{s}{(s-1)(s-5)(s+2)} = \frac{A}{s-1} + \frac{B}{s-5} + \frac{C}{s+2}$$

$$A(s-5)(s+2) + B(s-1)(s+2) + C(s-1)(s-5) = s$$

$\underline{s=5}$: $B \cdot 4 \cdot 7 = 5$; $B = \frac{5}{28}$

$\underline{s=-2}$: $C \cdot (-3)(-7) = -2$ $C = \frac{-2}{21}$

$\underline{s=1}$: $A \cdot (-4)(3) = 1$ $A = -\frac{1}{12}$

②

$$Y = -\frac{1}{12} \cdot \frac{1}{s-1} + \frac{5}{28} \cdot \frac{1}{s-5} - \frac{2}{21} \frac{1}{s+2}$$

$$y(t) = \mathcal{L}^{-1}\{Y\}$$

$$= \boxed{-\frac{1}{12}e^t + \frac{5}{28}e^{5t} - \frac{2}{21}e^{-2t}}$$

4. **[4 points]** Compute $\mathcal{L}\{1\}$ directly from the definition of the Laplace transform. To next sheet

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1\, dt = \left(-\frac{1}{s}e^{-st}\right)_{t=0}^{t\to\infty}$$

$$= \lim_{t\to\infty} \underbrace{\cancel{\left(-\frac{1}{s}e^{-st}\right)}}_{\to 0 \text{ for } s>0} - \left(-\frac{1}{s}e^0\right)$$

$$= 0 + \frac{1}{s} = \boxed{\frac{1}{s}} \text{ for } s > 0.$$

5. **[20 points]** Solve the IVP: $y''+4y = \cancel{f(t)}\ u_6(t)$, $y(0)=y'(0)=0$, where $\cancel{f(t) = \begin{cases}\cos 2t & \text{if } 0\le t<2\pi \\ 0 & \text{if } t\ge 2\pi\end{cases}}$.

$$\left[s^2Y - sy(0) - y'(0)\right] + 4Y = \mathcal{L}\left\{\left(1-u_{2\pi}(t)\right)\cos 2t\right\}$$

$$(s^2+4)Y = \mathcal{L}\{\cos(2t)\} - \mathcal{L}\{u_{2\pi}(t)\cos 2t\}$$

mult of 2π

$$(s^2+4)Y = \frac{s}{(s^2+4)} - e^{-2\pi s}\,\mathcal{L}\{u_{2\pi}(t)\cos(2(t-2\pi)\ \cancel{+4\pi})\}$$

$$(s^2+4)Y = \frac{s}{s^2+4} - e^{-2\pi s}\,\mathcal{L}\{\cos(2t)\}$$

$$(s^2+4)Y = \frac{s}{s^2+4} - \frac{e^{-2\pi s}\cdot s}{s^2+4}$$

$$Y = \frac{s}{(s^2+4)^2}\left(1-e^{-2\pi s}\right).$$

$$\left[s^2Y - sy(0) - y'(0)\right] + 4Y = \frac{e^{-6s}}{s}$$

$$Y = \frac{e^{-6s}}{s(s^2+4)}$$

$$y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{e^{-6s}}{s(s^2+4)}\right\} = u_6(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\}\Bigg|_{t-6}$$

$$\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$$

$$A(s^2+4) + (Bs+C)s = 1$$

$s=0$: $A\cdot 4 = 1$; $A = \frac{1}{4}$

$s=2i$: $\left.\begin{array}{l}(B(2i)+C)(2i) = 1 \\ -4B + 4Ci = 1\end{array}\right\}$ $B = -\frac{1}{4}$, $C = 0$.

$$= u_6(t)\,\mathcal{L}^{-1}\left\{\frac{1}{4}\cdot\frac{1}{s} - \frac{1}{4}\cdot\frac{s}{s^2+4}\right\}\Bigg|_{t-6}$$

$$= u_6(t)\left(\frac{1}{4} - \frac{1}{4}\cos(2t)\right)\Bigg|_{t-6}$$

$$= u_6(t)\left(\frac{1}{4} - \frac{1}{4}\cos(2(t-6))\right)$$

$$= \boxed{\frac{u_6(t)}{4}\left(1-\cos(2(t-6))\right)}$$