Math261 ~~Test 1~~ Test 2 **September 15, 2014**

Name: Solutions

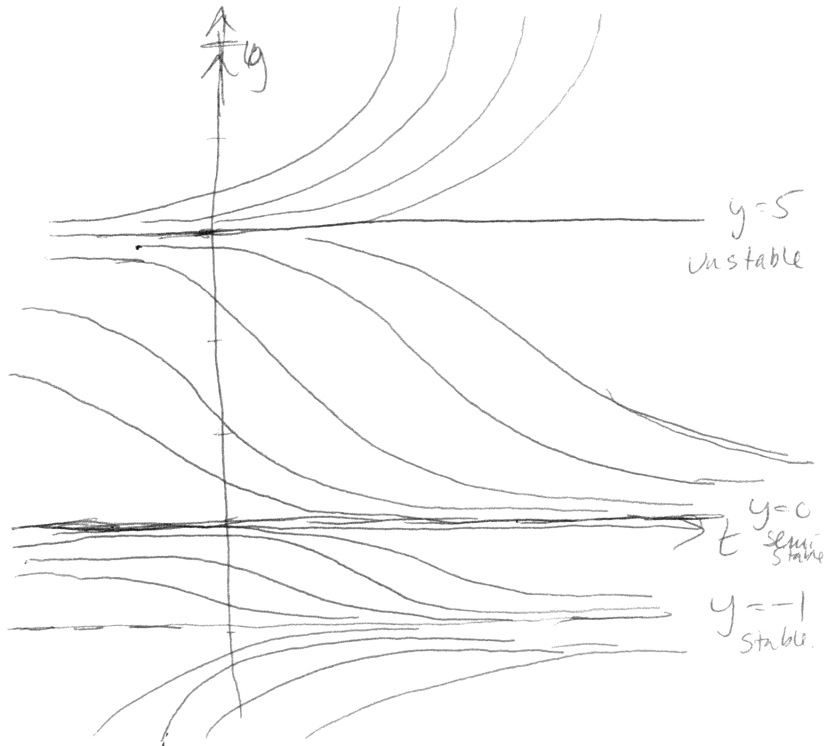
Directions: Show all work. No credit for answers without work.

1. Give a qualitative analysis of the following differential equation: $y' = y^3(y^2 - 4y - 5)$. That is, identify the equilibrium solutions and classify each as stable, semi-stable, or unstable. Include a sketch of typical solutions with a phase diagram.

$y' = y^3(y-5)(y+1)$

Equilibrium Solns:

$y=0,\ y=-1,\ y=5.$



From the chart:

$y=5$: unstable

$y=0$: semi stable

$y=-1$: stable

2. Solve the following differential equation: $6x^2y^2 + (e^y + ye^y + 4x^3y)y' = 0$.

$6x^2y^2\,dx + (e^y + ye^y + 4x^3y)dy = 0$

$\Rightarrow$ $M_y = 12x^2y$, $N_x = 12x^2y$, so the equ is exact

$\Rightarrow$ Impose $\psi_x = M$:

$\psi = \int 6x^2y^2\,dx$

$= 2x^3y^2 + h(y)$

$\Rightarrow$ Impose $\psi_y = N$.

$\frac{\partial}{\partial y}\left[2x^3y^2 + h(y)\right] = (e^y + ye^y + 4x^3y)$

$\cancel{4x^3y} + h'(y) = e^y + ye^y + \cancel{4x^3y}$

$h'(y) = (1+y)e^y$

$h(y) = \int (1+y)e^y\,dy$ $\quad u = 1+y \quad v = e^y$ $\quad du = dy \quad dv = e^y dy$

$= (1+y)e^y - \int e^y\,dy$

$= (1+y)e^y - e^y + C$

$= ye^y + C$

So $\psi = 2x^3y^2 + ye^y + C$

Soln: $\boxed{2x^3y^2 + ye^y = C}$

3. Solve the following IVP: $y'' + 4y' - 12y = 0$ with $y(0) = -1$ and $y'(0) = 1$.

$r^2 + 4r - 12 = 0$

$(r+6)(r-2) = 0$

$r_1 = -6,\ r_2 = 2$

$y = c_1 e^{-6t} + c_2 e^{2t}$

$y' = -6c_1 e^{-6t} + 2c_2 e^{2t}$

$\underline{y(0) = -1}$:

$-1 = c_1 + c_2$

$\underline{y'(0) = 1}$:

$1 = -6c_1 + 2c_2$

$+2 = -2c_1 - 2c_2$

$1 = -6c_1 + 2c_2$

$3 = -8c_1$

$c_1 = -\frac{3}{8}$

$c_2 = -1 - c_1 = -1 + \frac{3}{8} = -\frac{5}{8}$

$$\boxed{y = -\frac{3}{8} e^{-6t} - \frac{5}{8} e^{2t}}$$

4. Find the general solution to $y^{(5)} + 4y^{(4)} + 4y^{(3)} = 0$.

$r^5 + 4r^4 + 4r^3 = 0$

$r^3(r^2 + 4r + 4) = 0$

$r^3(r+2)^2 = 0$

$r = 0$, mult 3 | $r = -2$ mult 2

~~$r=0, r=4$~~

~~so~~

$y_1 = e^{0t} = 1$

$y_2 = te^{0t} = t$

$y_3 = t^2 e^{0t} = t^2$

$y_4 = e^{-2t}$

$y_5 = te^{-2t}$

Gen soln:

$$\boxed{y = c_1 + c_2 t + c_3 t^2 + c_4 e^{-2t} + c_5 te^{-2t}}$$

5. Find the general solution to $y'' - 10y' + 34y = te^t$.

(1) Solve associated homogeneous eqn:

$$r^2 - 10r + 34 = 0$$

$$r = \frac{10 \pm \sqrt{100 - 4(34)}}{2}$$

$$= 5 \pm \sqrt{25 - 34}$$

$$= 5 \pm \sqrt{-9} = 5 \pm 3i.$$

$$y_1 = e^{(5+3i)t} = e^{5t}(\cos(3t) + i\sin(3t))$$

$$y = c_1 e^{5t}\cos(3t) + c_2 e^{5t}\sin(3t).$$

(2). te^t

$e^t + \cancel{te^t}$

$\cancel{e^t}$

$$Y = Ae^t + Bte^t = (A + Bt)e^t.$$

$$Y' = Be^t + (A + Bt)e^t$$

$$= ((A+B) + Bt)e^t$$

$$Y'' = Be^t + ((A+B) + Bt)e^t$$

$$= ((A+2B) + Bt)e^t.$$

$$\begin{aligned} &((A+2B) + Bt)e^t \\ -10&[((A+B) + Bt)e^t] \\ +34&[(A + Bt)e^t] = te^t \end{aligned}$$

$$((25A - 8B) + 25Bt)e^t = te^t$$

Set ~~terms~~ like terms Equal:

$$25B = 1 \quad ; \quad B = \frac{1}{25}$$

$$25A - 8B = 0$$

$$25A - \frac{8}{25} = 0$$

$$A = \frac{8}{(25)^2} = \frac{8}{625}$$

$$Y = \left(\frac{8}{625} + \frac{1}{25}t\right)e^t$$

So the general soln is

$$\boxed{y = e^{5t}(c_1\cos(3t) + c_2\sin(3t)) + \left(\frac{8}{625} + \frac{1}{25}t\right)e^t}$$

6. Given that $y_1 = t^{-1}$ is a solution to $t^2y'' + 3ty' + y = 0$ for $t > 0$, find another solution y_2 that forms a fundamental set of solutions with y_1.

Reduction of order:

$$y = y_1 v = t^{-1} v$$

$$\cancel{y' = y_1 v +}$$

$$y' = y_1 v' + y_1' v = t^{-1}v' - t^{-2}v$$

$$y'' = y_1 v'' + y_1' v' + y_1' v' + y_1'' v$$

$$= y_1 v'' + 2y_1' v' + y_1'' v$$

$$= t^{-1}v'' - 2t^{-2}v' + 2t^{-3}v$$

Plug into diff equ:

$$t^2\left[t^{-1}v'' - 2t^{-2}v' + 2t^{-3}v\right]$$

$$+3t\left[t^{-1}v' - t^{-2}v\right]$$

$$+\left[t^{-1}v\right] = 0$$

$$tv'' + (-2+3)v' + \cancelto{0}{(2t^{-1} - 3t^{-1} + t^{-1})}\,v = 0$$

$$tv'' + v' = 0$$

Let $w = v'$.

$$tw' + w = 0$$

$$w' + \frac{1}{t}w = 0 \quad ; \quad \mu = e^{\int \frac{1}{t}dt} = e^{\ln t} = t$$

$$tw' + w = 0$$

$$\frac{d}{dt}\left[tw\right] = 0$$

$$tw = \int 0\,dt = c_1$$

$$w = \frac{c_1}{t}$$

$$v' = \frac{c_1}{t} \quad ; \quad v = \int \frac{c_1}{t}\,dt = c_1 \ln t + c_2$$

So $y = y_1 v = t^{-1}(c_1 \ln t + c_2)$

$$= \frac{c_1 \ln t + c_2}{t}$$

Choose $c_1 = 1$ and $c_2 = 0$ to get

$$\boxed{y_2 = \frac{\ln(t)}{t}}.$$

7. Show that $y_1 = \cos(t)$ and $y_2 = \sin(t)$ form a fundamental set of solutions to $y'' + y = 0$.

We show $W(y_1, y_2) \neq 0$.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix} = \cos^2(t) - (-\sin^2(t)) = 1.$$