Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[2 parts, 15 points each]** Solve the following explicitly.

(a) $\frac{dy}{dx} = x - 2y$

$$y' + 2y = x$$

$$\mu = e^{\int 2dx} = e^{2x}$$

$$e^{2x}y' + 2e^{2x}y = xe^{2x}$$

$$\frac{d}{dx}\left[e^{2x}y\right] = xe^{2x}$$

$$e^{2x}y = \int xe^{2x}dx$$

Integrate by parts

$u = x \quad v = \frac{1}{2}e^{2x}$

$du = dx \quad dv = e^{2x}dx$

$$e^{2x}y = \frac{x}{2}e^{2x} - \int \frac{1}{2}e^{2x}dx$$

$$e^{2x}y = \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C$$

$$\boxed{y = \frac{x}{2} - \frac{1}{4} + Ce^{-2x}}$$

(b) $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$.

1

$$\frac{dy}{dx} = \frac{\frac{1}{x^2}}{\frac{1}{x^2}}\left(\frac{x^2+y^2}{2xy}\right)$$

Homogeneous, $\frac{dy}{dx} = \frac{1+(\frac{y}{x})^2}{2(\frac{y}{x})}$

Let $v = \frac{y}{x}$, so $y = vx$, $\frac{dy}{dx} = x\frac{dv}{dx} + v$.

$$x\frac{dv}{dx} + v = \frac{1+v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{1+v^2}{2v} - \frac{v(2v)}{2v}$$

$$x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

2

$$\int \frac{2v}{1-v^2}dv = \int \frac{1}{x}dx$$

$u = 1-v^2$

$du = -2v\,dv$

$$\int -\frac{1}{u}du = \ln|x| + C$$

$$-\ln|1-v^2| = \ln|x| + C$$

$$-\left(\ln\left|1-\frac{y^2}{x^2}\right| + \ln|x|\right) = C$$

$$\ln\left|x\left(1-\frac{y^2}{x^2}\right)\right| = C$$

$$x\left(1-\frac{y^2}{x^2}\right) = C$$

$$1 - \frac{y^2}{x^2} = \frac{C}{x}$$

3

$$x^2 - y^2 = Cx$$

$$y^2 = x^2 - Cx$$

$$\boxed{y = \pm\sqrt{x^2 - Cx}}$$

2. **[2 parts, 15 points each]** Solve the following IVPs explicitly.

(a) $\dfrac{dy}{dt} = 2y^2 - 8$ with $y(0) = -3$.

$$\frac{dy}{dt} = 2(y^2-4)$$

$$\int \frac{1}{(y-2)(y+2)}\,dy = \int 2\,dt$$

$$\frac{A}{y-2} + \frac{B}{y+2} = \frac{1}{(y-2)(y+2)}$$

$$A(y+2) + B(y-2) = 1$$

$\underline{y=2}$: $A = \frac{1}{4}$

$\underline{y=-2}$: $B = -\frac{1}{4}$

$$\int \frac{1}{4}\cdot\frac{1}{y-2} - \frac{1}{4}\frac{1}{y+2}\,dy = 2t + C$$

$$\frac{1}{4}\ln|y-2| - \frac{1}{4}\ln|y+2| = 2t + C$$

$$\frac{1}{4}\ln\left|\frac{y-2}{y+2}\right| = 2t + C$$

$$\ln\left|\frac{y-2}{y+2}\right| = 8t + C$$

$$\frac{y-2}{y+2} = Ce^{8t}$$

$\underline{y(0)=-3}$: $\dfrac{-5}{-1} = C\cdot 1$; $C = 5$

$$\frac{y-2}{y+2} = 5e^{8t}$$

$$y - 2 = 5ye^{8t} + 10e^{8t}$$

$$y(1-5e^{8t}) = 2 + 10e^{8t}$$

$$\boxed{y = \frac{2+10e^{8t}}{1-5e^{8t}}}$$

(b) $\dfrac{dy}{dt} = \dfrac{\tan t}{y}$ with $y(0) = -1$. Hint: to solve $\int \tan t\,dt$, note $\tan t = \frac{\sin t}{\cos t}$ and use a substitution.

$$y\frac{dy}{dt} = \tan t$$

$$\int y\,dy = \int \tan t\,dt$$

$$\frac{y^2}{2} = -\ln|\cos(t)| + C$$

$\underline{y(0)=-1}$: $\dfrac{(-1)^2}{2} = -\cancel{\ln|1|}^{\,0} + C$; $C = \frac{1}{2}$

$$\frac{y^2}{2} = -\ln|\cos(t)| + \frac{1}{2}$$

$$y^2 = 1 - 2\ln|\cos(t)|$$

$$y^2 = 1 - \ln|\cos^2 t|$$

$$y^2 = 1 - \ln(\cos^2(t))$$

$$y = \pm\sqrt{1-\ln(\cos^2(t))}$$

Since $y(0) = -1$, choose

$$\boxed{y = -\sqrt{1-\ln(\cos^2(t))}}$$

3. A person wishes to finance a \$30,000 car with a loan that has an annual interest rate of %4. Assume that the loan payment is continuous and interest is compounded continuously.

   (a) **[8 points]** Let $B(t)$ be the balance of the loan (in dollars) at time t (in years), and let k be the annual payment rate. Write a differential equation for $B(t)$.

$$\frac{dB}{dt} = (0.04)B - k$$

$$\frac{dB}{dt} = \frac{1}{25}B - k$$

less space

   (b) **[8 points]** Solve the differential equation.

$$\frac{dB}{dt} = \frac{1}{25}(B - 25k)$$

$$\int \frac{1}{B-25k}\,dB = \int \frac{1}{25}\,dt$$

$$\ln|B - 25k| = \frac{1}{25}t + C$$

$$B - 25k = C e^{\frac{1}{25}t}$$

$$B(0) = 3\cdot 10^4: \qquad 3\cdot 10^4 - 25k = C\cancel{e}^{1}$$

$$B = 25k + (3\cdot 10^4 - 25k)e^{\frac{1}{25}t}$$

   (c) **[4 points]** Assuming the loan must be paid in full in 5 years, compute the annual payment rate, and convert it to a monthly rate.

Impose $B(5) = 0$:

$$0 = 25k + (3\cdot 10^4 - 25k)e^{\frac{1}{25}\cdot 5}$$

$$25k(e^{\frac{1}{5}} - 1) = 3\cdot 10^4 \cdot e^{\frac{1}{5}}$$

$$k = \frac{3\cdot 10^4 \cdot e^{\frac{1}{5}}}{25(e^{\frac{1}{5}} - 1)}$$

$k = \boxed{\$\,6619.99 \text{ per year or } \$\,551.67 \text{ per month}}$

4. **[4 parts, 5 points each]** If possible, apply the existence and uniqueness theorems to the following differential equations. ~~Be careful to show all relevant work~~. On the basis of these theorems, what can you conclude?

   (a) $(\sqrt{t}-2)\dfrac{dy}{dt}+(t-6)y=\ln(8-t)$ with $y(3)=5$

$$\frac{dy}{dt} + \frac{t-6}{\sqrt{t}-2}\,y = \frac{\ln(8-t)}{\sqrt{t}-2}.$$ Linear first order.

~~The solut~~ $\boxed{\text{There is a unique solution on } (2,8).}$

   (b) $\dfrac{dy}{dt}=\dfrac{2t}{\cos y}$ with $y(0)=0$

Non linear. $f(t,y)=\dfrac{2t}{\cos y}$ $\quad \dfrac{\partial f}{\partial y} = 2t(\sec y)(\tan y).$

Since f and $\dfrac{\partial f}{\partial y}$ are continuous when $\cos y \neq 0$, we conclude $\boxed{\text{the solution exists and is unique}}$.

   (c) $\dfrac{dy}{dx}=x\sqrt{y}$ with $y(4)=0$

Non linear. $f(x,y)=xy^{1/2}$ $\quad \dfrac{\partial f}{\partial y}=\frac{1}{2}xy^{-\frac{1}{2}}=\dfrac{x}{2\sqrt{y}}.$

Since f is continuous at $(0,4)$ but $\dfrac{\partial f}{\partial y}$ is not continuous at $(0,4)$, we conclude $\boxed{\text{a solution exists}}$. (Uniqueness is not guaranteed.)

   (d) $\dfrac{dy}{dx}=y\sqrt{x}$ with $y(4)=0$

Linear. $\dfrac{dy}{dx} - \sqrt{x}\,y = 0.$

Since $-\sqrt{x}$ is continuous on $(0,\infty)$ and 0 is continuous everywhere, we conclude ~~the solution~~ ~~there~~ that $\boxed{\text{there is a unique solution on } (0,\infty).}$