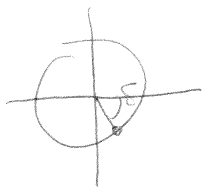
Name: Solutions

1. **[2 parts, 2 points each]** Convert the following functions to the form $R\cos(\omega_0 t - \delta)$.

   (a) $4\cos(2t) - 3\sin(2t)$

$R = \sqrt{4^2 + (-3)^2} = \sqrt{25} = 5$

Exact: $5\cos\left(2t - \tan^{-1}\left(-\frac{3}{4}\right)\right)$



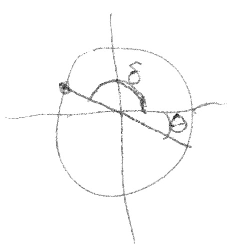
$\delta = \tan^{-1}\left(\frac{-3}{4}\right)$

Approx: $5\cos(2t + 0.6435)$

   (b) $-7\cos(t) + 2\sin(t)$

$R = \sqrt{(-7)^2 + 2^2} = \sqrt{53}$

$\theta = \tan^{-1}\left(-\frac{2}{7}\right)$



$\delta = \tan^{-1}\left(-\frac{2}{7}\right) + \pi$

Exact:

$\sqrt{53}\cos\left(t - \left(\tan^{-1}\left(-\frac{2}{7}\right) + \pi\right)\right)$

Approx:

$7.28\cos(t - 2.863)$

2. **[1 point]** An object of mass m (kg) is attached to a spring with spring constant k (kg/s^2). The system is damped with damping constant γ (kg/s). The system is critically damped if and only if m, γ, and k satisfy a certain equation. What is this equation? (Hint: if you do not have this memorized, derive it directly from the differential equation that models spring/mass systems.)

$mu'' + \gamma u' + ku = 0$

$mr^2 + \gamma r + k = 0$

$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$

Critically Damped: One real root of multiplicity 2. So

$$\boxed{\begin{array}{c}\gamma^2 - 4mk = 0 \\ \text{or} \\ \gamma = \sqrt{4mk}\end{array}}$$

3. A mass of 250 grams stretches a spring 8 cm. The system is undamped. Initially, the mass is pushed up(ward) a distance of 2 cm from its equilibrium position and released. with downward velocity of 10 cm/s.

   (a) **[3 points]** Find the position $u(t)$ of the spring at time t. Express u in cm and t in s.

Units: g, cm, s.

- $mu'' + \gamma u' + ku = 0$
- $m = 250$ g
- $\gamma = 0$ (undamped)
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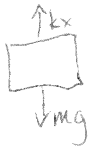

$mg = kx$; $g = 9.8\ m/s^2 = 980\ cm/s^2$

$$k = \frac{mg}{x} = \frac{(250)(980)}{8} = 30625\ g/s^2$$

$$250u'' + 30625u = 0$$

$$250r^2 + 30625 = 0$$

$$r = \pm\sqrt{\frac{245}{2}}\, i = \pm 7\sqrt{\frac{5}{2}}\, i \approx \pm 11.07t$$

$$u = c_1\cos\left(\sqrt{\frac{245}{2}}\,t\right) + c_2\sin\left(\sqrt{\frac{245}{2}}\,t\right)$$

$$u' = -\sqrt{\frac{245}{2}}\,c_1\sin\left(\sqrt{\frac{245}{2}}\,t\right) + \sqrt{\frac{245}{2}}\,c_2\cos\left(\sqrt{\frac{245}{2}}\,t\right)$$

$\underline{u(0) = -2}$: $-2 = c_1\cos(0) + \cancel{c_2\sin(0)}$ $c_1 = -2$

$\underline{u'(0) = 0}$: $\cancel{10}\ 0 = -\cancel{\sqrt{\frac{245}{2}}\,c_1\sin(0)} + \sqrt{\frac{245}{2}}\,c_2\cos(0)$ $c_2 = \sqrt{\frac{200}{245}}$

$$\boxed{u = -2\cos\left(\sqrt{\frac{245}{2}}\,t\right) + \sqrt{\frac{200}{245}}\sin\left(\sqrt{\frac{245}{2}}\,t\right)}$$

   (b) **[2 points]** Determine the maximum distance of the mass from its equilibrium position and the time when it first reaches this position. Hint: first express $u(t)$ in the form $u(t) = R\cos(\omega_0 t - \delta)$.

$$u = -2\cos\left(\sqrt{\frac{245}{2}}\,t\right) + \frac{2}{7}\sqrt{10}\sin\left(\sqrt{\frac{245}{2}}\,t\right)$$

$$u \approx -2\cos(11.07t) + 0.904\sin(11.07t)$$

$$R = \sqrt{(-2)^2 + \left(\sqrt{\frac{200}{245}}\right)^2} = \sqrt{4 + \frac{200}{245}} = \sqrt{\frac{236}{49}} = \frac{2}{7}\sqrt{59} \approx 2.195$$

$$\delta = \tan^{-1}\left(\frac{\frac{2}{7}\sqrt{10}}{-2}\right) + \pi = \tan^{-1}\left(-\frac{1}{7}\sqrt{10}\right) + \pi \approx 2.717$$

$$u \approx 2.195\cos\left(7\sqrt{\frac{5}{2}}\,t - 2.717\right) = 2.195\cos(11.07t - 2.717)$$

Max distance $= R \approx \boxed{2.195\text{ cm}}$

First time: $2.195 = 2.195\cos\left(7\sqrt{\frac{5}{2}}\,t - 2.717\right)$

$$1 = \cos\left(7\sqrt{\frac{5}{2}}\,t - 2.717\right)$$

$$7\sqrt{\frac{5}{2}}\,t - 2.717 = (\text{integer})\cdot\pi$$

$$7\sqrt{\frac{5}{2}}\,t = 2.717$$

$$\boxed{t \approx 0.245\text{ s}}$$