Name: Solutions

Directions: Show all work. No credit for answers without work.

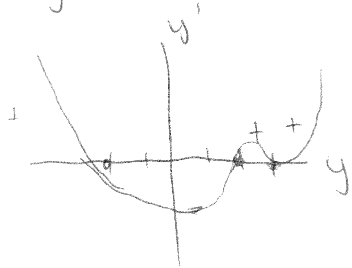
1. Give qualitative analysis of the following autonomous differential equations. That is, determine the equilibrium solutions, classify each as stable, unstable, or semistable, and sketch the solutions. Include a phase line.

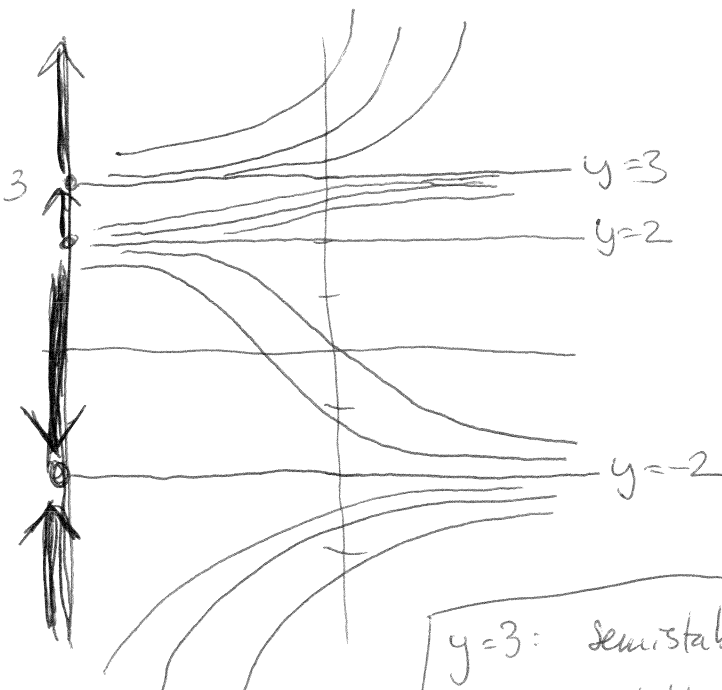
   (a) **[2 points]** $\frac{dy}{dt} = (y-3)^2(y^2-4)$

$y=3$ or $(y+2)(y-2)=0$

$y=3$ or $y=-2$ or $y=2$

$(y-3)^2(y+2)(y-2)$



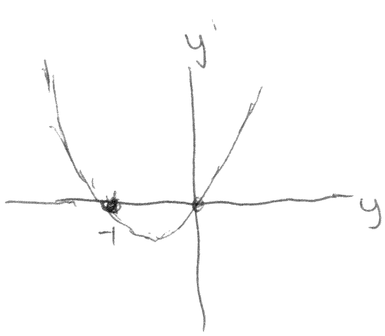


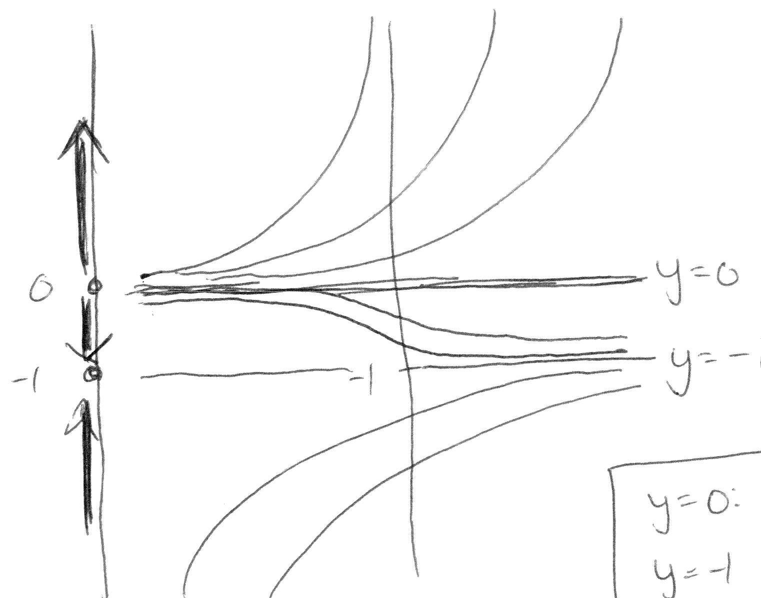
$y=3$: semistable
$y=2$: unstable
$y=-2$: stable

   (b) **[2 points]** $\frac{dy}{dt} = (y+1)\tan y$ for $-\pi/2 < y_0 < \pi/2$

$y+1=0$ or $\tan y = 0$, $-\pi/2 < y < \pi/2$

$y=-1$ or $y=0$





$y=0$: unstable
$y=-1$ stable

2. **[1 point]** Fill in the blank: under certain continuity assumptions, the equation $M(x, y) + N(x, y)y' = 0$ is exact if and only if $M_y = N_x$.

3. Solve the following exact differential equations.

   (a) **[2.5 points]** $(3x^2 + 2y^2) + (4xy + 6y^2)y' = 0$

$\psi_x = M$:

$\psi = \int 3x^2 + 2y^2 \, dx$

$= x^3 + 2y^2 x + h(y)$

$\psi_y = N$:

$\frac{\partial}{\partial y}\left[x^3 + 2y^2 x + h(y)\right] = 4xy + 6y^2$

$\cancel{4xy} + h'(y) = \cancel{4xy} + 6y^2$

$h'(y) = 6y^2$

$h(y) = \int 6y^2 dy = 2y^3 + C$

$\psi = x^3 + 2y^2 x + 2y^3 + C$

$\boxed{x^3 + 2y^2 x + 2y^3 = C}$

   (b) **[2.5 points]** $(1 + ye^{xy})dx + (2y + xe^{xy})dy = 0$

$\psi_x = M$:

$\psi = \int 1 + ye^{xy} \, dx$

$= x + e^{xy} + h(y)$

$\psi_y = N$:

$\frac{\partial}{\partial y}\left[x + e^{xy} + h(y)\right] = 2y + xe^{xy}$

$0 + \cancel{xe^{xy}} + h'(y) = 2y + \cancel{xe^{xy}}$

$h'(y) = 2y$

$h = y^2 + C$

$\psi = x + e^{xy} + y^2 + C$

$\boxed{x + e^{xy} + y^2 = C}$